Augmenting Social Influence of Uncertain Seeds via Probabilistic Link Insertion

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ABSTRACT

The emergence of link recommendation systems has triggered a line of research on strategic link insertion to enhance information diffusion in social networks. Existing literature assumes a seed set where all seed users are deterministically activated at the start of the campaign. However, uncertain seeding is being increasingly prevalent and can be used to model more general scenarios like users' defaulting behavior or discount-based marketing. To investigate how to augment the influence of uncertain seeds by link recommendation, we formulate a problem named *influence maximization with augmentation for uncertain seeds* (IMAUS), which aims to insert k edges incident to the uncertain seeds so as to maximize the influence of the given seeds. Due to the NP-hardness of the problem and the non-submodularity of the objective function, solving IMAUS is technically challenging. To address this, we resort to the sandwich strategy and propose two submodular bounding functions for the optimization objective. To overcome the #P-hardness of the bounding functions computation, we provide two unbiased estimators for the bounding functions via non-trivial usage of reverse influence sampling and devise greedy algorithms equipped with several principled accelerating techniques to return $(1-1/\mathrm{e}-\varepsilon)$-approximations for maximizing the bounding functions. With the above design, we instantiate the sandwich framework in a joint baking manner to reduce repeated sampling. Extensive experiments on 6 real-world datasets are conducted to validate the effectiveness and efficiency of the proposed methods. Specifically, our algorithm consistently produces a higher influence increment than the baselines and is able to return a size-100 edge set for a billion-size graph within 10 minutes.

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The source code, data, and/or other artifacts have been made available at https://github.com/chenxlong3/IMAUS.

1 INTRODUCTION

Information diffusion is a fundamental topic in social network analysis, where users are intricately connected, creating complex webs of relationships that enable the vast exchange of information, ideas, and opinions. Consequently, grasping and leveraging the underlying mechanisms of influence has become vital across various fields [20], including political campaigns and marketing [15]. A crucial research problem in this domain is influence maximization (IM) [27], which involves determining the most effective subset of nodes in a social network to serve as the initiators of influence spread, consequently maximizing the expected number of influenced nodes. The seminal work by Kempe et al. [27] presents IM as a submodular optimization problem, sparking extensive research that addresses various aspects of the IM problem [4, 6–8, 18, 21, 24, 32, 41, 44, 45]. These studies primarily focus on a static network topology.

Driven by the emergence of link recommendation systems such as Facebook's "People You May Know" and X's "Who to Follow," along with the extensive study of link recommendations for various objectives (e.g., minimizing polarization [1, 22, 55], maximizing fairness [47]), an increasing number of researchers investigate the impact of adding links to enhance influence spread [5, 9, 14, 25, 28, 50, 54]. However, existing work assumes a set of deterministic seeds. That is, the seed nodes are initially active and will certainly propagate the influence.

A more realistic and general setting is to treat the activation of seeds as an uncertain event. For example, Yang et al. [52, 53] consider providing discounts for users. Each user has a probability to become activated and proceed to propagate the influence, which is related to the offered discount. On the other hand, Tong et al. [46] incorporate activation failure probabilities of the seeds, which is able to model users' potential default behavior. Link recommendation for such a general scenario is yet to be discussed despite its potential for a wide spectrum of applications.

Applications. Conventional recommendation strategies [9, 11, 14] risk inefficacy when seed users fail to post promotional content [40, 46] (e.g., due to contractual defaults or platform's shadow banning [31]), and post-activation recommendation is not so profitable as users may exhibit limited engagement with the seeds' prior posts. While existing work tackles such failure by adaptively seeding new users [40, 46], such a *reactive* strategy inherently suffers from delay, as it requires a sequential process of observation and reseeding. Studying recommendations for uncertain seeds enables more robust proactive influence promotion. On the other hand, e-commerce platforms (e.g., Amazon, Shopify) frequently use limited-time discounts [52, 53] to incentivize user participation. Integrating

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link recommendations with such discount strategies can amplify influence propagation by aligning network effects with economic incentives, thereby enhancing marketing outcomes.

Challenges. Promoting the influence of uncertain seeds via strategic link insertion presents dual challenges in problem formulation and methodology design. The problem takes into account three kinds of uncertainties: probabilistic seed activation, stochastic influence propagation, and the inserted edges with diffusion probabilities. Such multifaceted uncertainty complicates objective function analysis. From the perspective of methodology, since the objective function is not submodular, the performance of a naive greedy algorithm does not provide any non-trivial guarantee. In addition, for the link insertion scenario, the search space is larger than solely selecting nodes from the network. Overall, designing an efficient algorithm with theoretical guarantees is technically challenging.

Contributions. To promote the influence of uncertain seed users via link recommendation in a social network, we formalize the *influence maximization with augmentation for uncertain seeds* (IMAUS) problem, proving its NP-hardness and demonstrating that the objective function is neither submodular nor supermodular. To tackle the non-submodularity, we propose two submodular bounding functions, which allow us to leverage the sandwich framework [16, 49] to obtain a data-dependent approximation ratio for the problem. Despite the #P-hardness of bounding functions computation, we propose efficient unbiased estimators via a non-trivial application of reverse influence sampling, which allows us to maximize the bounding functions by integrating OPIM-C [41] with greedy algorithms on the estimation functions. However, naïvely running greedy algorithm is computationally expensive due to the large search space, and a direct application of OPIM-C is infeasible as it is designed for selecting influential nodes but not edges. To address these issues, we further design some acceleration techniques like marginal gain maintenance and low probability edge pruning to optimize the greedy edge selection. As for the sampling procedure, we carefully modify OPIM-C to fit our scenario. Moreover, we observe that our estimate functions can be jointly optimized by sharing the samples, thereby realizing the sandwich framework in a joint-baking manner [23]. Finally, we conduct extensive experiments on 6 real-world datasets to demonstrate the effectiveness and efficiency of the proposed algorithm.

To summarize, we make the following contributions.

1. We propose a new problem termed IMAUS and analyze its properties thoroughly. To our knowledge, we are the first to study IM-oriented link recommendation for uncertain seed users. (§3)
2. To provide an algorithm with a non-trivial approximation ratio through a sandwich framework, we establish two submodular bounds of the objective function. (§4)
3. We design efficient estimators on the proposed bounds and design several pruning strategies to efficiently maximize the proposed bounds, with which we instantiate the sandwich algorithm in an efficient joint baking manner. (§5)
4. Extensive experiments are conducted on 6 real-world graphs to validate the effectiveness and efficiency of our approach. Specifically, the proposed algorithm returns solutions with higher influence than other baselines on all tested datasets and finishes within 10 minutes on Twitter, a dataset with over 1.5 billion edges. (§6)

Table 1: Frequently used notations

Notation	Description
$G = (V, E)$	A social network with node set V and edge set E
n, m	The number of nodes and edges in G
E_C	The set of all candidate edges
k	The number of edges to be selected
S, A	The seed set, and the edge set
$\mathbf{p}$	The activation probability vector (aka. configuration)
$A^\circ, A^\circ_L, A^\circ_U$	The optimal solutions for $\sigma(\cdot, \mathbf{p})$, $\sigma^L(\cdot, \mathbf{p})$ and $\sigma^U(\cdot, \mathbf{p})$, respectively
$\sigma(A, \mathbf{p})$	The augmented influence spread of configuration $\mathbf{p}$ after adding the edge set A
$\sigma^L(A, \mathbf{p}), \sigma^U(A, \mathbf{p})$	The lower bound and upper bound function of $\sigma(A, \mathbf{p})$
$\rho^L(A), \rho^U(A)$	$\rho^L(A) = \sigma^L(A, \mathbf{p}) - \sigma(\mathbf{p})$, $\rho^U(A) = \sigma^U(A, \mathbf{p}) - \sigma(\mathbf{p})$
$R, \mathcal{R}, \mathcal{R}_v$	A random RR set, a collection of RR sets, the RR sets in $\mathcal{R}$ that contains v
$\Gamma^L(A), \Gamma^U(A)$	The estimate of $\rho^L(A)$ and $\rho^U(A)$ (scaled by $\lvert\mathcal{R}\rvert / n$)

Due to the limited space, some proofs are omitted and could be found in the technical report [10] of this paper.

2 PRELIMINARIES

In this section, we introduce necessary preliminaries for the studied problem, including graph terminologies and the concept of reverse influence sampling, a widely used technique in conventional IM.

2.1 Influence Propagation

A social network is modeled as a directed graph $G = (V, E)$ with a node set V ($|V| = n$) and a directed edge set E ($|E| = m$). For an edge $\langle u, v \rangle \in E$, we call u an in-neighbor of v and v an out-neighbor of u. The in-neighbor (resp. out-neighbor) set of v is denoted by $N_{in}(v)$ (resp. $N_{out}(v)$). Each edge $\langle u, v \rangle$ is attached with an influence probability $p_{u,v} \in (0, 1]$, denoting the probability of u activating v.

Cascade model. In this paper, we adopt the widely-studied independent cascade (IC) model [4, 7, 27, 45]. The detail of influence propagation is illustrated as follows. At the initial timestamp 0, the chosen seed nodes are activated, leaving all other nodes inactive. When a node u first gets activated at timestamp i, it has a single opportunity to activate each node v of its inactive out-neighbors with probability $p_{u,v}$ at timestamp $i+1$, and it stays active for the duration of the propagation process. The diffusion process terminates when no additional nodes in the graph can be activated.

The information propagation process could be also described by *live edge* representation [27]. For the IC model, after removing each edge $\langle u, v \rangle \in E$ with probability $1 - p_{u,v}$, the remaining graph is referred to as a *realization* of G, which is denoted by ϕ. That is, for a particular $\phi \sim G$, $\Pr[\phi] = \prod_{e \in E_\phi} p_e \prod_{e \in E \setminus E_\phi} (1 - p_e)$. Let $I_\phi(S)$ be the number of nodes reachable from S in ϕ. The influence spread of a seed set S is defined as $\sum_{\phi \sim G} \Pr[\phi] I_\phi(S)$.

Influence of uncertain seeds. Probabilistic seeding is adopted to model consumer behaviors by previous literature [46, 52, 53]. Specifically, we assume a given probability vector $\mathbf{p} \in [0, 1]^n$ indicating the probability of each node being a deterministic seed and a node with non-zero entry in $\mathbf{p}$ is called an uncertain seed. Following prior work [52], we name $\mathbf{p}$ as a *configuration.* Mathematically, the probability that a subset $X \subseteq S$ is a deterministic seed set is computed by $\Pr[X] = \prod_{i \in X} \mathbf{p}_i \prod_{j \in S \setminus X} (1 - \mathbf{p}_j)$. Then considering ϕ as a realization of the uncertain graph G, the influence spread of a configuration $\mathbf{p}$ is defined as

$$\sigma(\mathbf{p}) = \sum_{\phi \sim G} \sum_{X \subseteq S} \Pr[\phi] \Pr[X] I_{\phi}(X). \tag{1}$$

The influence of a deterministic seed set S is denoted by $\sigma(\mathbf{1}_S)$, where $\mathbf{1}_S$ is the indicator vector with entries of 1 for $v \in S$ and 0 otherwise. Similarly, the influence of a singleton v is $\sigma(\mathbf{1}_v)$.

Augmented influence via link insertion. Given an edge set $A \subseteq (S \times V) \setminus E$, the augmented influence spread of the configuration $\mathbf{p}$ after adding edge set A into the network is defined as

$$\sigma(A, \mathbf{p}) = \sum_{\phi \sim G} \sum_{X \subseteq S} \sum_{T \subseteq A} \Pr[\phi] \Pr[X] \Pr[T] I_{\phi(T)}(X), \tag{2}$$

where $\Pr[T] = \prod_{\langle u,v \rangle \in T} p_{u,v} \prod_{\langle u,v \rangle \in A \setminus T} (1 - p_{u,v})$ and $\phi(T)$ is the resulting realization after adding the edge set T into ϕ. For further convenience, we use A_u^+ to denote the edges in A that start from node u, and A_v^- to denote the edges in A that end at node v. That is, $A_u^+ = \{\langle i,j \rangle \in A : i = u\}$ and $A_v^- = \{\langle i,j \rangle \in A : j = v\}$. For simplicity, we denote $\sigma(\emptyset, \mathbf{p})$ as $\sigma(\mathbf{p})$.

2.2 Reverse Influence Sampling

Borgs et al. [4] introduce a novel reverse influence sampling (RIS) framework for IM, which leverages sketch-based samples termed random *reverse reachable (RR) sets* to estimate seed set influence. A random RR set is generated through a two-step process:

(1) Select a node v from V uniformly at random.
(2) Sample a set R of the nodes in V, such that for any $u \in V$, the probability that it appears in R equals the probability that u can activate v in an influence propagation process.

Then the connection between the influence of a deterministic seed set S and a random RR sample is established as follows.

Lemma 2.1 ([4]). *For any seed set $S \in V$ and a random RR set R,*

$$\sigma(\mathbf{1}_S) = n \cdot \Pr[S \cap R \neq \emptyset].$$

Given a set $\mathcal{R}$ of random RR sets, we say that a RR set $R \in \mathcal{R}$ is *covered* by a node set S if $R \cap S \neq \emptyset$. Denote by $\mathcal{R}_S$ the RR sets covered by S. Then, $\frac{n}{|\mathcal{R}|} |\mathcal{R}_S|$ is an unbiased estimate of $\sigma(\mathbf{1}_S)$ according to Lemma 2.1. Based on RIS, Borgs et al. [4] propose a general framework for IM consisting of two steps. That is, we first (i) generate a set $\mathcal{R}$ of random RR sets, and then (ii) identify a node set S^* with the maximum coverage in $\mathcal{R}$ via a standard greedy algorithm. Utilizing RIS, the state-of-the-art IM algorithms [21, 24, 41, 44] have been proposed to reduce computation overheads.

Summary on notations. Throughout the paper, we denote the configuration by $\mathbf{p}$, the seed set by S, and the inserted edge set by A. $\sigma(A, \mathbf{p})$ is the influence of $\mathbf{p}$ after inserting A into the network. A random RR set is denoted by R, and $\mathcal{R}$ means a collection of random RR sets. Frequently used notations are summarized in Table 1.

3 PROBLEM FORMULATION

Conventional IMA problem aims to recommend deterministic seeds to ordinary nodes, while our problem extends it to the scenario of promoting the influence of uncertain seeds, whose formal definition is given by Definition 3.1.

Definition 3.1 (Influence Maximization with Augmentation for Uncertain Seeds (IMAUS)). Given a graph $G = (V, E)$, a set of probabilistic seeds S, a configuration $\mathbf{p} \in \mathbb{R}^n$ and a candidate edge set $E_C \subseteq (S \times V) \setminus E$, IMAUS aims to select a size-k edge set $A \subseteq E_C$ to maximize the augmented influence spread $\sigma(A, \mathbf{p})$.

For the time complexity analysis throughout this paper, we consider the worst-case scale of the candidate edge set, i.e., $|E_C| \in \Theta(|S|\, n)$.

Details on problem setting. Here we briefly discuss some key points of the problem setting. First, in contrast to deterministic recommendations [1, 13, 38, 54], the newly inserted edges are also probabilistic entities (influence probability), and the original network structure and diffusion probabilities remain fixed. This reflects real-world marketing scenarios where social relationships are observable and influence probabilities can be estimated from historical interactions [17, 39]. The advertiser may aim to maximize the social influence of the seed users by recommending them to others. However, the establishment of new connections (e.g., paid or sponsored) does not guarantee the success of influence propagation. Therefore, each inserted edge is also attached with a diffusion probability. Such an intervention does not affect the existing social relationship, as the recommendations normally do not affect current relationship in the real world. For the scenario where the sponsored recommendations may be refused, we can simply modify the probability to incorporate the acceptance probability (e.g., via simple multiplication), which can be done during the pre-processing step. Second, similar to prior studies [9, 11, 12, 14], we restrict candidate edges to those incident from the seed nodes. This restriction aligns with real-world incentives: seeds as campaign initiators are typically more willing to be promoted and gain followers to maximize exposure, whereas ordinary users may resist being recommended to others due to privacy concerns. In addition, recommending seed users to others is more likely to maximize the influence in some scenarios. For example, consider adding a new edge $\langle u, v \rangle$ pointing to v into the network with a fixed diffusion probability $p_{u,v}$. Let p_u^{ac} be the activation probability of u in the graph. Then the the activation probability increment of v is $p_u^{ac} \cdot p_{u,v}(1 - p_v^{ac})$. When there is only one seed or the seeds are deterministic, we have $p_u^{ac} \leq \mathbf{p}_s$, with s being the seed. In this case, the activation probability increment of v is maximized when the inserted edge is incident from s.

Computation hardness. Since the proposed problem is a generalization of the NP-hard IMA problem, we have the following theorem to state the computation hardness of IMAUS.

Theorem 3.2. *IMAUS is NP-hard and the computation of $\sigma(A, \mathbf{p})$ is #P-hard.*

Moreover, we show that $\sigma(\cdot, \mathbf{p})$ is neither submodular nor supermodular under the IC model, making it hard to approximate.

Lemma 3.3. *$\sigma(A, \mathbf{p})$ is monotone but neither submodular nor supermodular w.r.t. A under the IC model.*

Also, we prove that the objective function is not even weakly-submodular and thus, a greedy algorithm fails to provide any non-trivial approximation ratio.

Lemma 3.4. *$\sigma(A, \mathbf{p})$ is not weakly-submodular w.r.t. A.*

Discussion. The proposed IMAUS problem exhibits broad generalizability. On the one hand, when $\mathbf{p}$ satisfies that $\mathbf{p}_s = 1$ for all $s \in S$,

Algorithm 1 Sandwich Approximation Strategy

```
Input: A graph $G$, the budget $k$ and the parameters $\alpha_L, \alpha_U, \gamma$
Output: A size-$k$ edge set $A^*$
$A_U \leftarrow$ an $\alpha_U$-approximate solution to the upper bound function;
$A_L \leftarrow$ an $\alpha_L$-approximate solution to the lower bound function;
$A_H \leftarrow$ an arbitrary solution to the original function;
$\hat{\sigma}(\cdot, \mathbf{p}) \leftarrow$ a multiplicative $\gamma$-error estimate for $\sigma(\cdot, \mathbf{p})$
$A^* \leftarrow \arg\max_{A \in \{A_L, A_U, A_H\}} \hat{\sigma}(A, \mathbf{p})$
return $A^*$;
```

IMAUS becomes IMA [9, 12, 14], which aims to insert links pointing from deterministic seeds to ordinary ones to augment the influence spread. On the other hand, the general link recommendation for IM where no constraint is imposed on the candidate edges can also be modeled by IMAUS since the source of each candidate edge has a probability to be activated. While the exact activation probability of a node is usually intractable, lower-bound approximations (e.g., maximum influence probability path [6] or hop-based computation [42]) enable practical initialization for the probabilities and transform the general link recommendation problem to IMAUS.

4 SANDWICH STRATEGY

4.1 Overview

Due to the non-submodularity of the objective function, it is hard to solve IMAUS with a constant approximation ratio. Fortunately, a data-dependent approximation ratio could be obtained by using a sandwich strategy, described in Algorithm 1. Specifically, with a submodular lower bound $\sigma^L(\cdot, \mathbf{p})$ and a submodular upper bound $\sigma^U(\cdot, \mathbf{p})$ for the objective function $\sigma(\cdot, \mathbf{p})$, the algorithm first find an α_L-approximate solution A_L and an α_U-approximate solution A_U to the lower bound and upper bound, respectively. Then it computes a heuristic solution A_H to the original problem. We use a multiplicative γ-error estimate $\hat{\sigma}(\cdot, \mathbf{p})$ for $\sigma(\cdot, \mathbf{p})$ (i.e., $(1-\gamma)\sigma(A, \mathbf{p}) \leq \hat{\sigma}(A, \mathbf{p}) \leq (1+\gamma)\sigma(A, \mathbf{p})$) to compare the performance of A_L, A_U and A_H. Finally, the best among A_L, A_U and A_H is returned as the final solution. A well-established result [34] is that the solution A^* returned by the sandwich framework provides the following approximation guarantee.

$$\sigma(A^*, \mathbf{p}) \geq \max\left\{\frac{\sigma^L(A_L^\circ, \mathbf{p})}{\sigma(A^\circ, \mathbf{p})}\alpha_L, \frac{\sigma(A_U, \mathbf{p})}{\sigma^U(A_U, \mathbf{p})}\alpha_U\right\}\frac{1-\gamma}{1+\gamma}\sigma(A^\circ, \mathbf{p}), \quad (3)$$

where A_L° and A° are the optimal solutions for $\sigma^L(\cdot, \mathbf{p})$ and $\sigma(\cdot, \mathbf{p})$, respectively. Obviously, the sandwich framework relies on the bounds for the objective function. Next we present our design of monotone and submodular bounding functions for IMAUS.

4.2 Submodular Lower Bound

Intuitively, the non-submodularity comes from the fact that the activation probability of a candidate edge's source may be increased by the insertion of another edge. The rationale of designing the lower bound is that we only consider the influence increment of an edge $\langle s, v\rangle$ when the seed s is successfully initiated, ignoring the events where s is activated by other seeds. Following this, the lower bounding function is defined as

$$\sigma^L(A, \mathbf{p}) = \sum_{\phi\sim G}\sum_{X\subseteq S}\sum_{T\subseteq A} \Pr[\phi]\Pr[X]\Pr[T] I_{\phi(T_X)}(X), \quad (4)$$

where $T_X = \{\langle u, v\rangle \in T : u \in X\}$.

Lemma 4.1. *Given a configuration* $\mathbf{p}$ *and an edge set* A, $\sigma^L(A, \mathbf{p})$ *is monotone nondecreasing and submodular w.r.t.* A *under the IC model.*

4.3 Submodular Upper Bound

Since the activation probability of any node is upper bounded by 1, in order to design the submodular upper bounding function, we may assume that the inserted edges are all from deterministic seeds, which means that when an inserted edge $\langle u, v\rangle$ is live, v is deemed activated. Formally, we construct the upper bound by

$$\sigma^U(A, \mathbf{p}) = \sum_{\phi\sim G}\sum_{X\subseteq S}\sum_{T\subseteq A} \Pr[\phi]\Pr[X]\Pr[T] I_\phi(X \cup S_T), \quad (5)$$

where $S_T = \{v : \langle u, v\rangle \in T\}$. The monotonicity and submodularity are supported by Lemma 4.2.

Lemma 4.2. *Given a configuration* $\mathbf{p}$ *and an edge set* A, $\sigma^U(A, \mathbf{p})$ *is monotone nondecreasing and submodular w.r.t.* A *under the IC model.*

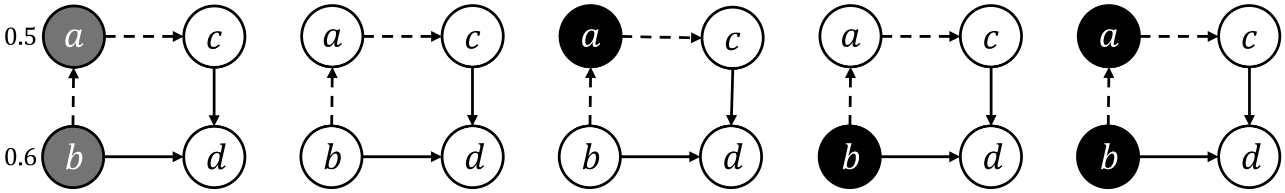
The hardness of computing the proposed bounding functions is given by Lemma 4.3.

Lemma 4.3. *Both computing* $\sigma^U(\cdot, \mathbf{p})$ *and* $\sigma^L(\cdot, \mathbf{p})$ *are #P-hard.*

Proof Sketch. Given an uncertain graph G, s-t reliability is defined as $\sum_{\phi\sim G}\Pr[\phi]\mathbb{I}(s \xrightarrow{\phi} t)$, where $\mathbb{I}(s \xrightarrow{\phi} t) = 1$ if s reaches t in the realization ϕ and 0 otherwise. Assuming we have an empty edge set A and a single uncertain seed s, then $\sigma^L(A, \mathbf{p}) = \sigma^U(A, \mathbf{p}) = \sigma(\mathbf{p})$. The s-t reliability from s to a certain target t can be computed as follows. We first compute $\sigma(\mathbf{p})$ in G. Then add a new node t' and an edge $\langle t, t'\rangle$ attached with probability 1 to the graph. The influence of s in the new graph is denoted by $\sigma'(\mathbf{p})$. Then the s-t reliability is simply $(\sigma'(\mathbf{p}) - \sigma(\mathbf{p}))/\mathbf{p}_s$. Since the s-t reliability computation is #P-complete [2], it naturally holds that the computation of $\sigma^L(A, \mathbf{p})$ and $\sigma^U(A, \mathbf{p})$ is #P-hard. □

Example 4.4. Consider the graph given in Figure 1a, consisting of four nodes (a, b, c and d) and two edges ($\langle b, d\rangle$ and $\langle c, d\rangle$). The seed set is set as $\{a, b\}$, with activation probability $\mathbf{p}_a = 0.5$ and $\mathbf{p}_b = 0.6$. Suppose an edge set $A = \{\langle b, a\rangle, \langle a, c\rangle\}$ is to be inserted into the network and the influence probability of each edge is fixed as 1. The original objective function $\sigma(A, \mathbf{p})$ can be simply computed by $0.2 \cdot 0 + 0.2 \cdot 3 + 0.3 \cdot 4 + 0.3 \cdot 4 = 3$. For the lower bound function, each inserted edge is only taken into account when the source of it is successfully activated as a seed. Therefore, in Figure 1c, $\langle a, c\rangle$ will not be inserted into the graph. Then the lower bounding function $\sigma^L(A, \mathbf{p}) = 0.2 \cdot 0 + 0.2 \cdot 3 + 0.3 \cdot 3 + 0.3 \cdot 4 = 2.7$. For the upper bound function, recall that for every live newly inserted edge, we assume the target node of it will be activated in that realization. As a result, for the case of Figure 1b, nodes a and c will be considered activated by the two inserted edges. Hence, the upper bound $\sigma^U(A, \mathbf{p}) = 0.2 \cdot 3 + 0.2 \cdot 3 + 0.3 \cdot 4 + 0.3 \cdot 4 = 3.6$.

The effectiveness of the sandwich framework essentially relies on tight submodular bounding functions. To provide an intuition on the tightness of the proposed bounds, we discuss equality conditions between them and the objective function: When the seeds are deterministically activated (i.e., $\mathbf{p} = \mathbf{1}_S$), we have $\sigma^L(A, \mathbf{1}_S) = \sigma(A, \mathbf{1}_S) = \sigma^U(A, \mathbf{1}_S)$. When the uncertain seeds cannot be activated by other nodes (i.e., every uncertain seed has no incoming edge or candidate edge), $\sigma^L(A, \mathbf{p}) = \sigma(A, \mathbf{p})$.



(a) Original Graph (b) $X = \emptyset$ $\Pr[X] = 0.2$ (c) $X = \{a\}$ $\Pr[X] = 0.2$ (d) $X = \{b\}$ $\Pr[X] = 0.3$ (e) $X = \{a, b\}$ $\Pr[X] = 0.3$

Figure 1: An example to illustrate the bounding functions.

Algorithm 2 JB-PIUS

```
Input: A graph $G$, the budget $k$, the error parameters $\varepsilon$ and $\gamma$
Output: A size-$k$ edge set $A^*$
$\theta_0 \leftarrow$ initial number of samples;
$\theta_{\max} \leftarrow$ the worst-case number of samples to return guaranteed solutions;
$i_{\max} \leftarrow \lceil \log_2 \frac{\theta_{\max}}{\theta_0} \rceil$
generate two collections of random RR sets $\mathcal{R}_1$ and $\mathcal{R}_2$ with $|\mathcal{R}_1| = |\mathcal{R}_2| = \theta_0$;
$\alpha_L \leftarrow 0$; $\alpha_U \leftarrow 0$;
for $i \leftarrow 1$ to $i_{\max}$ do
    if $\alpha_L < 1 - 1/\mathrm{e} - \varepsilon$ then
        $A_L \leftarrow$ select the edge set by Algorithm 3
        $\alpha_L \leftarrow$ compute the empirical guarantee for $\rho^L(\cdot)$;
    if $\alpha_U < 1 - 1/\mathrm{e} - \varepsilon$ then
        $A_U \leftarrow$ select the edge set by Algorithm 4
        $\alpha_U \leftarrow$ compute the empirical guarantee for $\rho^U(\cdot)$;
    if both $\alpha_L$ and $\alpha_U$ are larger than $1 - 1/\mathrm{e} - \varepsilon$ then
        break;
    double the size of $\mathcal{R}_1$ and $\mathcal{R}_2$ by generating new samples;
$A_H \leftarrow$ an arbitrary heuristic solution to the original function;
$\hat{\sigma}(\cdot, \mathbf{p}) \leftarrow$ a multiplicative $\gamma$-error estimate for $\sigma(\cdot, \mathbf{p})$;
$A^* \leftarrow \arg\max_{A \in \{A_L, A_U, A_H\}} \hat{\sigma}(A, \mathbf{p})$;
return $A^*$;
```

5 EFFICIENT IMPLEMENTATION

With the newly proposed submodular bounding functions, the key challenge lies in efficiently instantiating the sandwich framework. In this section, we begin by introducing the overall joint baking framework that uses the same set of samples to optimize both bounding functions (§5.1), and then detail the maximization of the bounding functions, consisting of efficient estimators (§5.2), refined greedy selection algorithms (§5.3) and a modified sampling schedule (§5.4). Finally, we put it together and provide theoretical analysis on the proposed algorithm (§5.5).

5.1 A Joint Baking Framework

For the purpose of efficient implementation, we instantiate the aforementioned sandwich framework in a joint baking manner [23], where the bounding functions are maximized using the same set of samples, which brings much efficiency by avoiding repeated sampling. The proposed algorithm is called JB-PIUS (**J**oint **B**aking algorithm to **P**romote the **I**nfluence of **U**ncertain **S**eeds), the pseudocode of which is given in Algorithm 2. Based on the idea of OPIM-C [41], two collections of RR sets will be sampled to compute the empirical approximation ratio of the bounding functions. JB-PIUS first determines the initial and worse-case sample size. In each iteration of the main loop, we maximize the bounding functions by running well-designed greedy algorithms on the sampled RR sets, and test whether the empirical guarantees (α_L and α_U) exceed $1 - 1/\mathrm{e} - \varepsilon$. We exit the loop if both α_L and α_U are larger than $1 - 1/\mathrm{e} - \varepsilon$. Otherwise, the sample size will be doubled with newly sampled RR sets. After the loop, we have two approximate solutions A_L and A_U, for the lower and upper bound function, respectively. Use A_H to denote any heuristic solution to the original problem. We will finally compare the quality of A_H, A_L and A_U with a multiplicative γ-error estimator and select the best one among them. Next we will delve into the details of obtaining approximate solutions for both bounding functions.

5.2 Bounding Functions Estimation

As the approximation ratio for submodular optimization mainly focuses on normalized functions, i.e., $f(\emptyset) = 0$, we will discuss the normalized bounding functions in the following, denoted as $\rho^L(A, \mathbf{p}) = \sigma^L(A, \mathbf{p}) - \sigma(\mathbf{p})$ and $\rho^U(A, \mathbf{p}) = \sigma^U(A, \mathbf{p}) - \sigma(\mathbf{p})$. In the context of a fixed configuration $\mathbf{p}$, we simplify the upper and lower bound functions as $\rho^L(A)$ and $\rho^U(A)$, respectively. The corresponding estimator functions are denoted by $\Gamma^L(A)$ and $\Gamma^U(A)$, which will be introduced later.

To efficiently maximize the bounding functions that are #P-hard to compute, we should first investigate how to estimate the bounding functions by using the sampled RR sets. For both the upper and lower bound estimation, the intuition is to compute the probability that a random RR set changes its state from *not covered* to *covered* by the given seed set. To estimate the normalized influence increment of a given edge set, we should first eliminate the effect of the existence of uncertain seeds. Under the setting of conventional IM, each RR set can be abstracted as a Bernoulli random variable equal to 0 or 1, and the initial utility value of each RR set is 1 as no RR set has been covered yet. However, in our case of uncertain seeds, each RR set is initially covered with a certain probability by the seeds. So in order to estimate the normalized functions, we should initialize the utility value of each RR set by computing the probability that an RR set is not covered by any seed node. Specifically, the initial utility value of an RR set R is defined as

$$\pi_0(R) = \prod\nolimits_{u \in R}(1 - \mathbf{p}_u), \tag{6}$$

which could be computed and stored during the sampling phase. Then we delve into the formulation of the estimators for the proposed bounding functions.

Lower bound estimation. Now consider the lower bound function, which only counts the contribution of the edges pointing from successfully activated seeds. That is, for any edge $\langle u, v\rangle \in A$, we only consider the insertion of it when u is successfully activated as a seed. Intuitively, for a particular RR set R, the event that R changes its state from uncovered to covered could be considered as the intersection of the following two events - (i) R is not covered by any successfully activated seed, and (ii) there is at least one successfully activated seed u, such that at least one edge in A_u^+ activates R. Following this intuition, we derive an unbiased estimator for the lower bound function, formalized by Lemma 5.1.

Lemma 5.1. *Let R be a random RR set, for any inserted edge set A, we have*

$$\rho^L(A) = n \cdot \mathbb{E}\left[(1 - \prod\nolimits_{u \in S \setminus R}(1 - \mathbf{p}_u w(u, R, A)))\pi_0(R)\right], \qquad (7)$$

where $w(u, R, A) = 1 - \prod_{\langle u,v\rangle \in A_u^+}(1 - p_{u,v}\mathbb{I}(v \in R))$.

Proof. Let $S_{T,X}$ be $\{v \in V : \langle u, v\rangle \in T \text{ and } u \in X\}$. Then by the rationale of RIS, we have

$$\begin{aligned}\rho^L(A) &= \sigma^L(A, \mathbf{p}) - \sigma(\mathbf{p})\\ &= \sum_{X \subseteq S, T \subseteq A} \Pr[X]\Pr[T] \sum_{v \in V} \Pr[(X \nrightarrow v) \wedge (S_{T,X} \rightarrow v)]\\ &= n \cdot \mathbb{E}_R\mathbb{E}_X\mathbb{E}_T[\mathbb{I}(R \cap X = \emptyset)\mathbb{I}(R \cap S_{T,X} \neq \emptyset)].\end{aligned}$$

Since $R \cap S_{T,X} \neq \emptyset$ indicates that at there is least one seed $u \in X$, such that at least one $\langle u, v\rangle \in A_u^+$ satisfies that $\langle u, v\rangle \in T$ and $v \in R$, we have

$$\begin{aligned}&\mathbb{I}(R \cap S_{T,X} \neq \emptyset)\\ =&1 - \prod_{u \in X} \prod_{\langle u,v\rangle \in A_u^+} (1 - \mathbb{I}(\langle u, v\rangle \in T)\mathbb{I}(v \in R))\\ =&1 - \prod_{u \in S}(1 - \mathbb{I}(u \in X)(1 - \prod_{\langle u,v\rangle \in A} (1 - \mathbb{I}(\langle u, v\rangle \in T)\mathbb{I}(v \in R)))).\end{aligned}$$

The correctness of the second equality could be easily verified by considering two events, $u \in X$ and $u \notin X$. Define $W_T(u, R, A) = (1 - \prod_{\langle u,v\rangle \in A_u^+}(1 - \mathbb{I}(\langle u, v\rangle \in T)\mathbb{I}(v \in R)))$ for convenience and $\mathbb{E}_T[W_T(u, R, A)] = w(u, R, A)$. We have

$$\begin{aligned}&\mathbb{I}(R \cap X = \emptyset)\mathbb{I}(R \cap S_{T,X} \neq \emptyset)\\ =&\prod_{u \in R} \mathbb{I}(u \notin X) \cdot (1 - \prod_{u \in S}(1 - \mathbb{I}(u \in X)W_T(u, R, A)))\\ =&\prod_{u \in R} \mathbb{I}(u \notin X) \cdot (1 - \prod_{u \in S \setminus R} (1 - \mathbb{I}(u \in X)W_T(u, R, A))).\end{aligned}$$

The first equality is obtained by the definition of $R \cap X \neq \emptyset$, and the second equality holds since if $R \cap X = \emptyset$, for every $u \in R$, u should not be in X, and if $R \cap X \neq \emptyset$, the result is 0, same for both side. By the linearity of expectation and the mutual independence of all seed and edge activations, we have

$$\begin{aligned}&\mathbb{E}_R\mathbb{E}_X\mathbb{E}_T\left[\prod_{u \in R} \mathbb{I}(u \notin X) \cdot (1 - \prod_{u \in S \setminus R} (1 - \mathbb{I}(u \in X)W_T(u, R, A)))\right]\\ =&\mathbb{E}_R\mathbb{E}_X\left[\prod_{u \in R} \mathbb{I}(u \notin X) \cdot (1 - \prod_{u \in S \setminus R} (1 - \mathbb{I}(u \in X)\mathbb{E}_T[W_T(u, R, A)]))\right]\\ =&\mathbb{E}_R\left[\prod_{u \in R} \mathbb{E}_X[\mathbb{I}(u \notin X)] \cdot (1 - \prod_{u \in S \setminus R} (1 - \mathbb{E}_X[\mathbb{I}(u \in X)]w(u, R, A)))\right]\\ =&\mathbb{E}_R[\pi_0(R) \cdot (1 - \prod\nolimits_{u \in S \setminus R}(1 - \mathbf{p}_u w(u, R, A)))].\end{aligned}$$

Hence the lemma holds. □

Define

$$\Gamma^L(A) = \sum\nolimits_{R \in \mathcal{R}}(1 - \prod\nolimits_{u \in S \setminus R}(1 - \mathbf{p}_u w(u, R, A)))\pi_0(R). \qquad (8)$$

Then with a collection $\mathcal{R}$ of random RR sets, $\frac{n\Gamma^L(A)}{|\mathcal{R}|}$ serves as an unbiased estimate for $\rho^L(A)$.

Upper bound estimation. For the upper bound function, all inserted edges are assumed to be pointed from deterministic seeds. Hence, for any inserted edge $\langle u, v\rangle$, RR sets containing v have an additional probability of $p_{u,v}$ to be covered. Then considering all edges in A, the probability of an RR set changes its state (from not covered to covered) is $(1 - \prod_{v \in R} \prod_{\langle u,v\rangle \in A_v^-}(1 - p_{u,v}))$. Taking $\pi_0(R)$ into account, the estimator of the upper bound is then formalized by Lemma 5.2.

Lemma 5.2. *Let R be a random RR set. For any inserted edge set A, we have*

$$\rho^U(A) = n \cdot \mathbb{E}\left[(1 - \prod\nolimits_{v \in R} \prod\nolimits_{\langle u,v\rangle \in A_v^-}(1 - p_{u,v}))\pi_0(R)\right]. \qquad (9)$$

Proof. Similar to the proof of lower bound estimation, we have

$$\begin{aligned}\sigma^U(A, \mathbf{p}) - \sigma(\mathbf{p}) &= \mathbb{E}_R\mathbb{E}_X\mathbb{E}_T[\mathbb{I}(R \cap X = \emptyset)\mathbb{I}(R \cap S_T \neq \emptyset)]\\ &= \mathbb{E}_R\left[\mathbb{E}_X[\mathbb{I}(R \cap X = \emptyset)]\mathbb{E}_T[\mathbb{I}(R \cap S_T \neq \emptyset)]\right]\\ &= \mathbb{E}_R\left[\mathbb{E}_X\left[\prod\nolimits_{u \in R}\mathbb{I}(u \notin X)\right]\mathbb{E}_T[\mathbb{I}(R \cap S_T \neq \emptyset)]\right]\\ &= \mathbb{E}_R[\pi_0(R)\mathbb{E}_T[\mathbb{I}(R \cap S_T \neq \emptyset)]].\end{aligned}$$

The second equality is due to the fact that the activation of uncertain seeds is independent of the existence of inserted edges, and the third equality comes from the definition of $R \cap X = \emptyset$. The fourth equality is obtained by the linearity of expectation and the independence of each seed's activation. Now we analyze the computation of $\mathbb{E}_T[\mathbb{I}(R \cap S_T \neq \emptyset)]$.

$$\begin{aligned}\mathbb{E}_T[\mathbb{I}(R \cap S_T \neq \emptyset)] &= \mathbb{E}_T[1 - \prod_{v \in R}\mathbb{I}(v \notin S_T)]\\ &= \mathbb{E}_T[1 - \prod_{v \in R} \prod_{\langle u,v\rangle \in A_v^-} (1 - \mathbb{I}(\langle u, v\rangle \in T))]\\ &= 1 - \prod_{v \in R} \prod_{\langle u,v\rangle \in A_v^-} (1 - p_{u,v})\end{aligned}$$

Putting together completes the proof. □

Defining

$$\Gamma^U(A) = \sum\nolimits_{R \in \mathcal{R}}(1 - \prod\nolimits_{v \in R} \prod\nolimits_{\langle u,v\rangle \in A_v^-}(1 - p_{u,v}))\pi_0(R), \qquad (10)$$

$\frac{n\Gamma^U(A)}{|\mathcal{R}|}$ brings an unbiased estimate for $\rho^U(A)$ after sampling a collection of random RR sets $\mathcal{R}$.

It is worth noting that both the lower and upper bound can be estimated within the same sample space, which builds the foundation of the joint baking framework to eliminate redundant sampling.

Discussion on degeneracy. As discussed in Section 4, when the seeds are deterministic (i.e., $\mathbf{p} = \mathbf{1}_S$), $\rho^L(A) = \rho^U(A) = \sigma(A, \mathbf{1}_S) - \sigma(\mathbf{1}_S)$. Under this condition, $\pi_0(R) = \mathbb{I}(R \cap S = \emptyset)$. Then following Lemma 5.2, we can easily derive that

$$\begin{aligned}&\sigma(\{\langle u, v\rangle\}, \mathbf{1}_S) - \sigma(\mathbf{1}_S)\\ =&n \cdot \mathbb{E}\left[\mathbb{I}(v \in R)(1 - (1 - p_{u,v}))\mathbb{I}(R \cap S = \emptyset)\right]\\ =&np_{u,v}\mathbb{E}\left[\mathbb{I}(v \in R)\mathbb{I}(R \cap S = \emptyset)\right] = p_{u,v}(\sigma(\mathbf{1}_{S \cup \{v\}}) - \sigma(\mathbf{1}_S)),\end{aligned}$$

which is the foundation of AIS [9] algorithm (Lemma 4.1 in their paper) for the IMA problem.

5.3 Greedy Selection

With the unbiased estimators above, a $(1 - 1/e)$ approximation on $\Gamma^L(\cdot)$ (resp. $\Gamma^U(\cdot)$) yields a $(1 - 1/e - \varepsilon)$-approximation for the maximization of $\rho^L(\cdot)$ (resp. $\rho^U(\cdot)$) given enough samples [44]. Now we show that such a guarantee could be obtained via a simple greedy algorithm by proving that both $\Gamma^L(\cdot)$ and $\Gamma^U(\cdot)$ are monotone non-decreasing and submodular.

Lemma 5.3. *$\Gamma^L(A)$ and $\Gamma^U(A)$ are monotone non-decreasing and submodular w.r.t. A.*

However, if we compute $\Gamma^L(\cdot)$(or $\Gamma^U(\cdot)$) naively in every iteration, the greedy algorithm takes up to $O(k\,|S|\,n \sum_{R\in\mathcal{R}} |R|)$ time, which is prohibitive. So we incorporate the following techniques to accelerate the implementation.

Lazy update. We first employ a widely-adopted acceleration technique in submodular maximization, namely CELF [32]. Specifically, for a submodular function $\Gamma(A)$, if $\Gamma(e_1|A) \geq \Gamma(e_2|A')$ for $A' \subseteq A$, $\Gamma(e_1|A) \geq \Gamma(e_2|A)$ naturally holds, where $\Gamma(e|A) = \Gamma(A \cup \{e\}) - \Gamma(A)$. Hence, if the marginal gain of an edge remains the largest after updating, there is no need to evaluate the marginal gain of the rest candidate edges.

Marginal gain maintenance. To reduce redundant computation, a natural idea is to maintain the marginal gain of each candidate element. However, such maintenance requires rigorous analysis. Let $\langle u^*, v^* \rangle$ be the edge to be inserted into A and $\langle u', v' \rangle$ be any other edge in $E_C \setminus A$. We have the following lemma for $\Gamma^L(\cdot)$.

Lemma 5.4. *Use $\Gamma^L(e|A)$ to denote $\Gamma^L(A \cup \{e\}) - \Gamma^L(A)$. We have*

$$\begin{aligned}&\Gamma^L(\langle u', v' \rangle \mid A \cup \{\langle u^*, v^* \rangle\})\\ =&\Gamma^L(\langle u', v' \rangle \mid A) - \sum_{R\in\mathcal{R}_{v^*}\cap\mathcal{R}_{v'}\setminus\mathcal{R}_{u'}\setminus\mathcal{R}_{u^*}} \pi(R)\cdot a\cdot b,\end{aligned}$$

where $\pi(R) = \pi_0(R)\prod_{u\in S\setminus R}(1 - \mathbf{p}_u w(u, R, A))$, $a = \mathbf{p}_{u'} p_{u',v'} \cdot \frac{1-w(u',R,A)}{1-\mathbf{p}_{u'}\,w(u',R,A)}$ and

$$b = \begin{cases} p_{u^*,v^*} & \text{, if } u' = u^* \\ \mathbf{p}_{u^*} p_{u^*,v^*} \frac{1-w(u^*,R,A)}{1-\mathbf{p}_{u^*}\,w(u^*,R,A)} & \text{, else.} \end{cases}$$

Then for the greedy algorithm on $\Gamma^L(\cdot)$, we first initialize the marginal gain of each edge as $\Delta(\langle u, v \rangle) = \mathbf{p}_u p_{u,v} \sum_{R\in\mathcal{R}_v\setminus\mathcal{R}_u} \pi_0(R)$, where $\mathcal{R}_v = \{R \in \mathcal{R} : v \in R\}$. In each iteration, after selecting the edge with the highest marginal gain, denoted by $\langle u^*, v^* \rangle$, we traverse the RR sets in $\mathcal{R}_v \setminus \mathcal{R}_u$. For each node v in the RR set, we update the marginal gain of each $\langle u, v \rangle \in E_C$ that ends at v with $u \notin R$ by subtracting $\Delta(\langle u, v \rangle)$ by $\pi(R) \cdot a \cdot b$, with a and b defined by Lemma 5.4. After that, we need to update the value of $\pi(R)$ by

$$\pi(R) \leftarrow \pi(R) \cdot \frac{1 - \mathbf{p}_{u^*} w(u^*, R, A \cup \{\langle u^*, v^* \rangle\})}{1 - \mathbf{p}_{u^*} w(u^*, R, A)}. \tag{11}$$

Next we turn to the marginal gain maintenance for $\Gamma^U(\cdot)$, which is characterized by Lemma 5.5.

Algorithm 3 Edge Selection for Lower Bound Maximization

```
Input: $E_C, \mathcal{R}, \mathbf{p}, k$
Output: A size-$k$ edge set $A^*$
$A^* \leftarrow \emptyset$; $S_v \leftarrow \emptyset$ for $v \in V$;
for $\langle u, v \rangle \in E_C$ do
    Add $u$ to $S_v$;
    $\Delta(\langle u, v \rangle) \leftarrow \mathbf{p}_u p_{u,v} \cdot \sum_{R\in\mathcal{R}_v\setminus\mathcal{R}_u} \pi_0(R)$;
initialize $\pi(R) \leftarrow \pi_0(R)$ for all $R \in \mathcal{R}$;
initialize $w(u, R) \leftarrow 0$ for all $u \in S$ and $R \in \mathcal{R}$;
for $i = 1$ to $k$ do
    $\langle u^*, v^* \rangle \leftarrow \arg\max_{\langle u,v\rangle\in E_C\setminus A^*} \Delta(\langle u, v \rangle)$;
    $A^* \leftarrow A^* \cup \{\langle u^*, v^* \rangle\}$;
    for $R \in \mathcal{R}_{v^*} \setminus \mathcal{R}_{u^*}$ do
        for $v \in R$ do
            for $u \in S_v \setminus R$ do
                if $u = u^*$ then
                    $a \leftarrow \mathbf{p}_u p_{u,v} \frac{(1-w(u,R))}{(1-\mathbf{p}_u\,w(u,R))}$;
                    $b \leftarrow p_{u^*,v^*}$;
                    $\Delta(\langle u, v \rangle) \leftarrow \Delta(\langle u, v \rangle) - \pi(R) \cdot a \cdot b$;
                else
                    $a \leftarrow \mathbf{p}_u p_{u,v} \frac{(1-w(u,R))}{(1-\mathbf{p}_u\,w(u,R))}$;
                    $b \leftarrow \mathbf{p}_{u^*} p_{u^*,v^*} \frac{(1-w(u^*,R))}{(1-\mathbf{p}_{u^*}\,w(u^*,R))}$
                    $\Delta(\langle u, v \rangle) \leftarrow \Delta(\langle u, v \rangle) - \pi(R) \cdot a \cdot b$;
        $\pi(R) \leftarrow \pi(R)/(1 - \mathbf{p}_{u^*} w(u^*, R))$;
        $w(u^*, R) \leftarrow w(u^*, R) + (1 - w(u^*, R))p_{u^*,v^*}$;
        $\pi(R) \leftarrow \pi(R)(1 - \mathbf{p}_{u^*} w(u^*, R))$;
return $A^*$;
```

Lemma 5.5. *Use $\Gamma^U(e|A)$ to denote $\Gamma^U(A \cup \{e\}) - \Gamma^U(A)$. Then*

$$\begin{aligned}&\Gamma^U(\langle u', v' \rangle \mid A \cup \{\langle u^*, v^* \rangle\})\\ =&\Gamma^U(\langle u', v' \rangle \mid A) - \sum_{R\in\mathcal{R}_{v^*}\cap\mathcal{R}_{v'}} p_{u^*,v^*} p_{u',v'} \cdot \pi(R),\end{aligned}$$

where $\pi(R) = \pi_0(R) \cdot \prod_{v\in R} \prod_{\langle u,v\rangle\in A_v^-} (1 - p_{u,v})$.

With Lemma 5.5, we detail the greedy selection algorithm for the upper bound as follows. The marginal gain of every edge $\langle u, v \rangle \in V$ is initialized as $p_{u,v} \cdot \sum_{R\in\mathcal{R}_v} \pi_0(R)$. After selecting the edge with the highest marginal gain $\langle u^*, v^* \rangle$, we traverse the RR sets containing v^* (denoted by $\mathcal{R}_{v^*}$). Given an RR set $R \in \mathcal{R}_{v^*}$, for each node $v \in R$, we update the marginal gain of the candidate edges pointing to v by subtracting $p_{u,v} p_{u^*,v^*} \pi(R)$. Then we need to update the utility value of each $R \in \mathcal{R}_{v^*}$ by

$$\pi(R) \leftarrow \pi(R)(1 - p_{u^*,v^*}). \tag{12}$$

Repeating such a procedure for k times gives the final solution.

Low probability edge pruning for upper bound maximization. For the upper bounding function, a convenient property is that given two edges $e_1 = \langle u_1, v \rangle$ and $e_2 = \langle u_2, v \rangle$ with $p_{u_1,v} \geq p_{u_2,v}$, we have $\Gamma^U(A \cup \{e_1\}) \geq \Gamma^U(A \cup \{e_2\})$. Hence, if e_1 is not yet selected, we do not need to update the marginal gain of e_2. Specifically, we first pack the edges pointing to the same node into an array and sort them according to the edge probability and store the largest marginal gain among the candidate edges pointing to v as $\Delta(v)$. After selecting $\langle u^*, v^* \rangle$, we update $\Delta(v)$ for each node v in every RR set $R \in \mathcal{R}_{v^*}$. In this way, we only maintain the marginal gain of nodes instead of edges.

Algorithm 4 Edge Selection for Upper Bound Maximization

```
Input: $E_C, \mathcal{R}, \mathbf{p}, k$
Output: A size-$k$ edge set $A^*$
1:  $A^* \leftarrow \emptyset; S_v \leftarrow \emptyset$ for $v \in V$;
2:  for $\langle u, v\rangle \in E_C$ do
3:      Add $u$ to $S_v$;
4:  for $v \in V \setminus S$ do
5:      sort $S_v$ by descending order of $p_{u,v}$;
6:      $u \leftarrow S_v.\text{top}(); \Delta(v) \leftarrow p_{u,v} \cdot \sum_{R\in\mathcal{R}_v} \pi_0(R)$;
7:  initialize $\pi(R) \leftarrow \pi_0(R)$ for all $R \in \mathcal{R}$;
8:  for $i = 1$ to $k$ do
9:      $v^* \leftarrow \arg\max_v \Delta(v); u^* \leftarrow S_{v^*}.\text{pop}()$;
10:     $A^* \leftarrow A^* \cup \{\langle u^*, v^*\rangle\}$;
11:     for $R \in \mathcal{R}_{v^*}$ do
12:         for $v \in R$ do
13:             $u \leftarrow S_v.\text{top}(); \Delta(v) \leftarrow \Delta(v) - \pi(R) \cdot p_{u^*,v^*} \cdot p_{u,v}$;
14:         $\pi(R) \leftarrow \pi(R)(1 - p_{u^*,v^*})$;
15:     $u \leftarrow S_{v^*}.\text{top}(); \Delta(v^*) \leftarrow \frac{p_{u,v^*}}{p_{u^*,v^*}} \cdot (1 - p_{u^*,v^*}) \cdot \Delta(v^*)$;
16: return $A^*$;
```

Putting it together. Above all, the greedy selection algorithms are given in Algorithm 3 and Algorithm 4, which basically follow the same paradigm. We maintain a marginal gain array $\Delta(\cdot)$ for edges and utility array $\pi(\cdot)$ for RR sets. In each iteration, we select the edge with the maximum marginal gain and then update the arrays correspondingly. The distinctions are two-fold: (i) the update mechanisms, and (ii) the pruning technique for the upper bound maximization. In the selection algorithm for lower bound, we maintain an extra two-dimensional array $w(u, R)$ to store $w(u, R, A)$ in each iteration, with all arrays updated dynamically across iterations (line 10-23 of Alg. 3). The selection algorithm for upper bound leverages the aforementioned pruning technique to maintain the gains of nodes instead of edges. The update of $\Delta(\cdot)$ and $\pi(\cdot)$ (line 11-14 of Alg. 4) follows Lemma 5.5 and Eq. (12). The time complexity results of them are given by Lemma 5.6.

Lemma 5.6. *Algorithm 3 returns a $(1 - 1/\mathrm{e})$-approximate solution to $\Gamma^L(\cdot)$ in $O(k\,|S| \sum_{R\in\mathcal{R}} |R|)$ time and Algorithm 4 returns a $(1 - 1/\mathrm{e})$-approximate solution to $\Gamma^U(\cdot)$ in $O(k \sum_{R\in\mathcal{R}} |R|)$ time.*

5.4 Sampling Schedule Details

Although the random variable attached with each RR set is no longer a Bernoulli random variable, it remains bounded in $[0, 1]$. Hence, the martingale inequalities underpinning OPIM-C [41] retain applicability. However, some necessary modifications need to be performed to ensure the theoretical guarantee, which are detailed below.

Empirical approximation ratio. Given the analogous forms of the bounds for $\rho^L(\cdot)$ and $\rho^U(\cdot)$, here we focus on $\rho^L(\cdot)$ for conciseness. Let A be the selected edge set and A_L° be the optimal solution for maximizing $\rho^L(\cdot)$. We construct the lower bound of $\rho^L(A)$ as

$$\rho_l^L(A) = \left(\left(\sqrt{\Gamma_2^L(A) + \frac{2\zeta}{9}} - \sqrt{\frac{\zeta}{2}}\right)^2 - \frac{\zeta}{18}\right) \cdot \frac{n}{\theta_2}. \tag{13}$$

and the upper bound of $\rho^L(A_L^\circ)$ as

$$\rho_u^L(A_L^\circ) = \left(\left(\sqrt{\frac{\Gamma_1^L(A)}{1 - 1/\mathrm{e}} + \frac{\zeta}{2}} + \sqrt{\frac{\zeta}{2}}\right)^2\right) \cdot \frac{n}{\theta_1}, \tag{14}$$

where $\Gamma_i^L(\cdot)$ means the estimate function $\Gamma^L(\cdot)$ using the collection $\mathcal{R}_i$ of random RR sets ($i \in \{1, 2\}$), and ζ's value is related to failure probability (detailed settings will be revealed later). Then the empirical approximation ratio could be computed by $\rho_l^L(A)/\rho_u^L(A_L^\circ)$.

Computing $\theta_{\max}$. To determine the worst-case number of samples, we first introduce the theoretical sample size to achieve the guarantee through Lemma 5.7.

Lemma 5.7 (Adapt from [44]). *Let OPT_L be the optimal value of $\rho^L(A)$. Alg. 3 (resp. Alg. 4) returns a $(1 - 1/\mathrm{e} - \varepsilon)$-approximation on $\rho^L(\cdot)$ (resp. $\rho^U(\cdot)$) with probability $1 - \delta/9$ if the number of RR sets θ satisfies*

$$\theta \geq \frac{2n\left((1 - 1/\mathrm{e})\sqrt{\log\frac{18}{\delta}} + \sqrt{(1 - 1/\mathrm{e})(\log\binom{|S|n}{k} + \log\frac{18}{\delta})}\right)^2}{\varepsilon^2 \mathrm{OPT}_L}.$$

Proof Sketch. Let $R_1, R_2, \ldots, R_\theta$ be a sequence of random RR sets and A be a set of inserted edges. Take $\rho^L(\cdot)$ as an example, and the proof for $\rho^U(\cdot)$ follows the same paradigm. Define $x_i = (1 - \prod_{u\in S\setminus R_i}(1 - \mathbf{p}_u w(u, R_i, A)))\pi_0(R_i)$, $p = \rho^L(A)/n$ and $M_i = \sum_{j=1}^{i}(x_j - p)$. We first prove that the sequence $M_1, M_2, \ldots, M_\theta$ induces a martingale (details are given in the technical report [10]), supporting the usage of the martingale concentration inequalities. Setting the failure probability to $\delta/9$ and taking a union bound on all $\binom{|S|n}{k}$ possible solutions yield the lemma. □

As OPT_L is intractable, we need to design a computationally easy way to identify the minimum possible value for OPT_L so as to obtain $\theta_{\max}$. Unlike conventional IM where the minimum possible influence is trivially k, such a lower bound depends on the problem instance. For example, consider all nodes in the network can be influenced with probability 1, indicating that inserting edges will not augment the influence. Therefore, we should first verify whether OPT_L is 0.

Without loss of generality, we suppose that the candidate edge set is not empty, since the problem is meaningless if no edge could be inserted into the network. $\mathrm{OPT}_L = 0$ indicates that no candidate edge is able to augment the influence of the given configuration $\mathbf{p}$. That is, the activation probability of each node cannot be amplified by the candidate edges. This occurs if every node v could be reached from a deterministic seed through a path whose edges' probabilities are all 1, or if there is no candidate edge that points to v. So before running our algorithm, we should first identify the deterministic seeds and run a BFS from them to check if adding edges can increase the influence or not. If $\mathrm{OPT}_L > 0$ is assured, we proceed to compute the lower bound of it.

Here we only consider the common case[1] where there exists at least one node having all incoming edges attached with influence probabilities smaller than 1. For any node v whose activation probability could be augmented, the minimum probability that this node will not be activated by others is $\prod_{u\in N_{in}(v)}(1-p_{u,v})$. By excluding the probability of v itself being an active seed and following the rule of the lower bounding function, the benefit of recommending a seed to v is at least

[1] A graph where every node has at least one in-neighbor with influence probability 1 can be converted to a DAG by Tarjan's algorithm. We can erase the loops and find a node whose incoming edges are with influence probability smaller than 1.

Table 2: Datasets ($K = 10^3$, $M = 10^6$, $B = 10^9$)

Name	n	m	Type	Avg. Degree
Epinions	75.9K	508.8K	Directed	13.4
Stanford	281K	2.3M	Directed	16.4
DBLP	317K	1.05M	Undirected	6.1
Orkut	3.1M	117.1M	Undirected	76.3
LiveJournal	4.8M	69M	Directed	28.5
Twitter	41.7M	1.5B	Directed	70.5

$$\kappa_v = \max_{s \in S \backslash N_{in}(v) \backslash \{v\}} \mathbf{p}_s (1 - \mathbf{p}_v) p_{s,v} \cdot \prod_{u \in N_{in}(v)} (1 - p_{u,v}). \tag{15}$$

By summing the top-k largest κ_v values across all $v \in V$, we obtain the minimum possible value of OPT_L, denoted by κ. Then the worst-case sampling size $\theta_{\max}$ is defined as

$$\theta_{\max} = \frac{2n \left((1 - 1/\mathrm{e}) \sqrt{\log \frac{18}{\delta}} + \sqrt{(1 - 1/\mathrm{e}) (\log \binom{|S|n}{k} + \log \frac{18}{\delta})} \right)^2}{\varepsilon^2 \kappa}. \tag{16}$$

5.5 Putting It Together

Now we put it together and give theoretical analysis on the proposed algorithm JB-PIUS. In Algorithm 2, we set $\theta_{\max}$ by Equation (16), and compute the empirical approximation ratio by Equation (13) and (14) with $\zeta = \frac{2\delta}{9 i_{\max}}$. Finally, for the evaluation of A_L, A_U and A_H, we conduct a $(\gamma, \delta/9)$-estimation on the augmented influence spread by utilizing the general stopping rules [56]. Combining these components, the approximation guarantee is established by the following theorem.

Theorem 5.8. *The solution A^* returned by Algorithm 2 satisfies*

$$\sigma(A^*, \mathbf{p}) \geq \max \left\{ \frac{\sigma^L(A_L^\circ, \mathbf{p})}{\sigma(A^\circ, \mathbf{p})}, \frac{\sigma(A_U, \mathbf{p})}{\sigma^U(A_U, \mathbf{p})} \right\} \frac{1 - \gamma}{1 + \gamma} (1 - 1/\mathrm{e} - \varepsilon) \sigma(A^\circ, \mathbf{p})$$

with probability at least $1 - \delta$.

The time complexity of JB-PIUS consists of RR sets sampling, selection and the final estimation stages. Combining the analysis of existing work [41] and Lemma 5.6, we have the following theorem regarding the running time of Algorithm 2.

Theorem 5.9. *With the failure probability $\delta \leq 1/2$, Algorithm 2 runs in $O(\frac{\tau(k|S|n+m) \ln(\binom{|S|n}{k}/\delta)}{\varepsilon^2 \mathrm{OPT}_L} + \frac{\tau(m+n) \ln(1/\delta)}{\gamma^2 \sigma(A^*,\mathbf{p})})$ expected time, where τ is the influence of the most influential singleton node.*

6 EXPERIMENTS

6.1 Experiment Settings

This section evaluates the empirical performance of the proposed algorithm. All experiments are conducted on a linux machine with Intel Xeon(R) 8377C@3.0GHz and 512GB RAM. All algorithms are implemented by C++ with O3 optimization. In our experiments, we run each method for 10 times and report the average results.

Datasets. We conduct experiments on 6 publicly available datasets [30, 33], the statistics of which are presented in Table 2. Among them, Twitter is one of the largest dataset used in the field of influence analysis with 1.5 billion edges. For undirected networks, we convert each edge $\langle u, v \rangle$ into two directed edges $\langle u, v \rangle$ and $\langle v, u \rangle$.

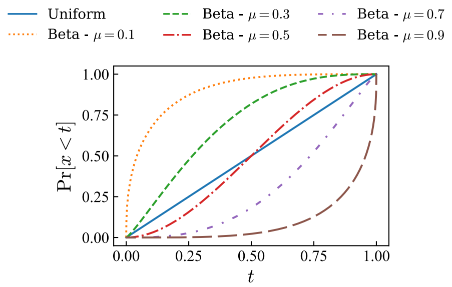


Figure 2: Cumulative distribution functions of all tested seed probability distributions.

Table 3: Parameters settings in our experiments.

Name	Default	Values
k	100	$\{5, 10, 20, 40, 60, 80, 100\}$
$\lvert S \rvert$	50	$\{10, 20, 30, 40, 50\}$
Seed Dist.	Uniform	{Uniform, Beta}
μ	-	$\{0.1, 0.3, 0.5, 0.7, 0.9\}$
ε	0.1	$\{0.1, 0.2, 0.3, 0.4, 0.5\}$
γ	0.05	$\{0.05, 0.1, 0.15\}$

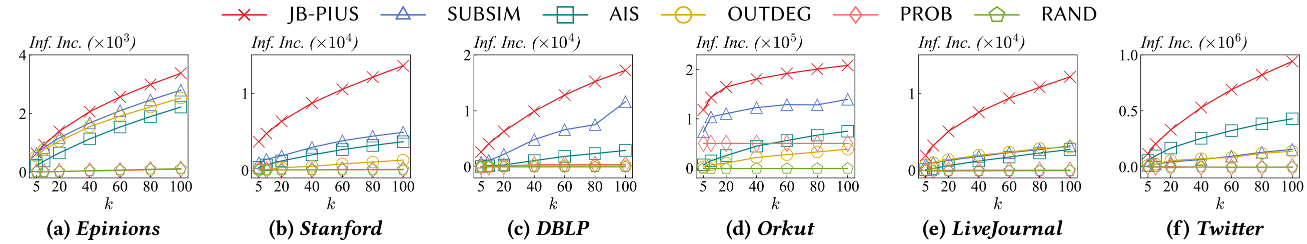
Implementation details. The sampling of each RR set is the same as the state-of-the-art sampling method SUBSIM [21]. We run JB-PIUS strictly following the approximation guarantee parameterized by corresponding ε and γ in each experiment.

Parameters. The propagation probability $p_{u,v}$ of each edge is set to be the inverse of v's in-degree, which is a widely-adopted setting [6, 24, 27, 41, 44, 45, 48] named weighted cascade (WC). The failure probability δ of our algorithm is fixed as $1/n$. Under the default setting, we set the seed set size $|S| = 50$, with the seeds selected by state-of-the-art IM algorithm [21], conforming the real scenario where we will select influential nodes as seed users. Following existing work [49], we set $\gamma = 0.05$ for a good balance between approximation quality and efficiency. Other default parameters include $k = 100$ and $\varepsilon = 0.1$. For the configuration $\mathbf{p}$, we assign a uniform random number in $(0, 1]$ to $\mathbf{p}_s$ for each $s \in S$. When studying the effect of different seeding probabilities, we employ Beta distribution with varying means μ for the probability assignment, whose cumulative distribution curves are shown in Figure 2. All parameter settings tested are summarized in Table 3.

To demonstrate the scalability of our algorithm, the candidate edge set E_C contains all possible edges, indicating that the size of E_C is $\Theta(|S|\, n)$. Every candidate edge $e = \langle u, v \rangle$ contains a seed node $u \in S$ and another node $v \in V \setminus \{u\}$. Following existing work [9], the propagation probability of a candidate edge $\langle u, v \rangle$ is assigned as $\frac{\bar{p}_u^{\mathrm{out}} + \bar{p}_v^{\mathrm{in}}}{2}$, where $\bar{p}_u^{\mathrm{out}} = \frac{\sum_{w \in N_{\mathrm{out}}(u)} p_{u,w}}{|N_{\mathrm{out}}(u)|}$ and $\bar{p}_v^{\mathrm{in}} = \frac{\sum_{w \in N_{\mathrm{in}}(v)} p_{w,v}}{|N_{\mathrm{in}}(v)|}$.

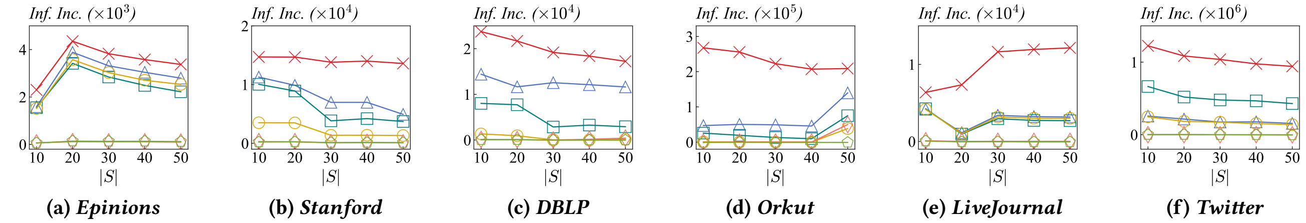
Algorithms. We compare JB-PIUS with the following methods.

- SUBSIM [21]. The state-of-the-art IM algorithm. We run the IM algorithm ($\varepsilon = 0.1$ an $\delta = 1/n$) to identify k nodes, and for each node, we select the edge pointing to it with the maximum attached probability.
- AIS [9]. An RIS-based method for IMA [12, 14]. The number of RR sets is set as the requirement to obtain a $(0.1, 1/n)$-estimation on $\sigma(\mathbf{p})$.



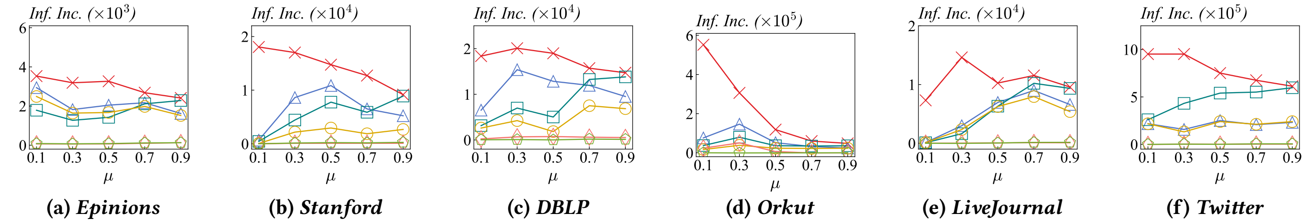
(a) ***Epinions*** (b) ***Stanford*** (c) ***DBLP*** (d) ***Orkut*** (e) ***LiveJournal*** (f) ***Twitter***

Figure 3: Influence increment with varying k.



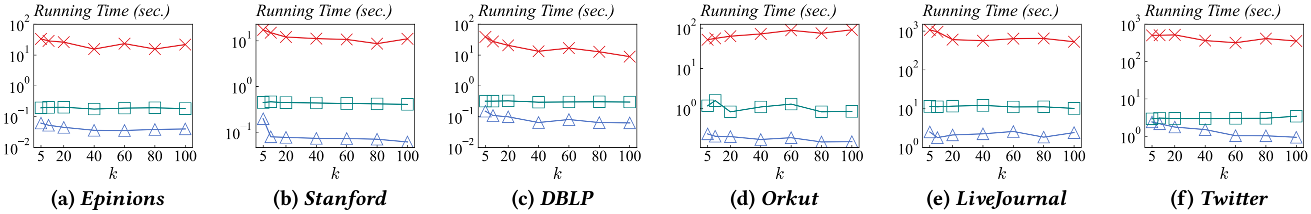
(a) ***Epinions*** (b) ***Stanford*** (c) ***DBLP*** (d) ***Orkut*** (e) ***LiveJournal*** (f) ***Twitter***

Figure 4: Influence increment with varying $|S|$.



(a) ***Epinions*** (b) ***Stanford*** (c) ***DBLP*** (d) ***Orkut*** (e) ***LiveJournal*** (f) ***Twitter***

Figure 5: Influence increment with varying μ.



(a) ***Epinions*** (b) ***Stanford*** (c) ***DBLP*** (d) ***Orkut*** (e) ***LiveJournal*** (f) ***Twitter***

Figure 6: Running time with varying k.

- OUTDEG. Identify top-k nodes with the highest out-degree, and for each node, we select the edge pointing to it with the maximum attached probability.
- PROB. Select top-k edges with the highest probability.
- RAND. Select k edges uniformly at random.

Evaluation metrics. We mainly investigate the following three metrics in our experiments to evaluate the effectiveness and efficiency of our algorithms.

- *Influence Increment.* The influence increment is computed by $\sigma(A, \mathbf{p}) - \sigma(\mathbf{p})$, where the influence spread is estimated by RIS [52] within a multiplicative error 0.01.
- *Running Time.* The running time of each algorithm is recorded for the efficiency test, excluding I/O of the graph.
- *Approximation Ratio.* Following existing work [49], we compute $\left(\frac{1-\gamma}{1+\gamma}\right)^2 \cdot (1 - 1/\mathrm{e} - \varepsilon) \cdot \frac{\hat{\sigma}(A_U,\mathbf{p})}{\hat{\sigma}^U(A_U,\mathbf{p})}$ as the lower bound of the data-dependent approximation ratio, which reflects the quality of the approximation.

6.2 Effectiveness Evaluation

Varying k. Figure 3 reports the resulting influence increment with varying k on all six datasets. It can be observed that JB-PIUS consistently outperforms the baselines in terms of influence increment, demonstrating JB-PIUS's effectiveness. Naive heuristics (i.e., OUTDEG, PROB and RAND) always fail to provide effective solutions to IMAUS with varying k. SUBSIM and AIS surpass heuristic approaches by targeting influential nodes, yet remain suboptimal compared to JB-PIUS. This highlights JB-PIUS 's superior ability to identify profitable edges to augment the influence.

Varying $|S|$. Figure 4 compares the influence increment of all methods w.r.t. different number of seeds $|S|$. The results indicate that JB-PIUS maintains consistent superiority to other baselines. An interesting observation is that the influence increment decreases with larger $|S|$ in some cases. This phenomenon arises because when more influential nodes are selected as seeds, recommending links to influential nodes will have a lower benefit given that such nodes

Table 4: Running time (sec.) breakdown of JB-PIUS

Procedure	Epinions	Stanford	DBLP	Orkut	LiveJournal	Twitter
Sampling	0.684	1.128	0.671	1.473	28.968	7.854
Selection	20.641	8.857	7.811	87.870	471.840	244.690
Estimation	0.585	1.106	0.436	1.186	30.945	103.046

may already be selected as seeds. But some datasets like LiveJournal show an upward trend. A possible reason is that larger $|S|$ expands candidate edge set, which may include more high-impact edges.

Varying seed probability distribution. To evaluate robustness under different seed activation probability distributions, we initialize seeding probabilities via Beta distributions with varying means μ. As presented in Figure 5, we can observe that JB-PIUS consistently achieves superior influence increment over other methods. Notably, when μ is large enough, AIS tends to have similar performance with JB-PIUS, since when the seeds are close to being deterministically activated, IMAUS will become IMA and AIS provides satisfactory results. In addition, we observe that the influence increment tends to decrease for most datasets when μ is large since high activation probabilities leave fewer nodes available for influence augmentation.

6.3 Efficiency Evaluation

Running time breakdown. Table 4 details the runtime breakdown of JB-PIUS, which completes execution in under 10 minutes even for the largest two datasets. The time cost is dominated by the selection phase since the ground set size is $\Theta(|S|\,n)$, unlike $O(n)$ for IM. The time cost for the solution quality estimation is at the same order of magnitude compared with that of the sampling phase for all datasets except Twitter.

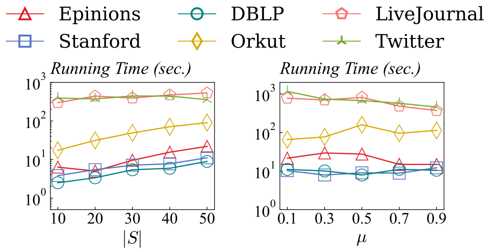
Varying k. Figure 6 reports the running time of the RIS-based algorithms for different k. When increasing k, the running time remains stable for all datasets. JB-PIUS is slower than SUBSIM and AIS. This overhead arises primarily from (i) a larger ground set in selection and (ii) an extra estimation step. Nevertheless, JB-PIUS finishes in acceptable time (within 10 minutes even for the billion-size dataset Twitter) and its asymptotic performance is similar to SUBSIM and AIS.

Varying $|S|$ and μ. Figure 7 reports JB-PIUS's runtime across varying $|S|$ and μ. For datasets except LiveJournal and Twitter, it only takes less than two minutes for our algorithm to finish. For LiveJournal and Twitter, our algorithm is able to finish within 20 minutes under all settings. We can see that the running time increases with larger $|S|$ since the candidate edge set is also enlarged, thereby increasing selection time. As for the efficiency with different μ, it can be observed that the algorithm's time cost does not present obvious trends with larger μ.

6.4 Sensitivity Evaluation

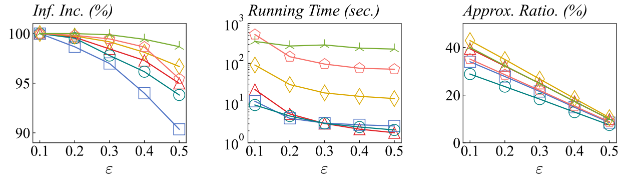
Now we turn to the sensitivity evaluation of the proposed algorithm w.r.t. different ε and γ.

Effect of ε. We vary ε from 0.1 to 0.5 to evaluate the influence increment, running time and approximation ratio, respectively. As shown in Figure 8, with the increase of ε, the resulting influence increment decreases for all datasets since the approximation ratios of the bounding functions decreases. Despite this, when $\varepsilon = 0.5$, JB-PIUS maintains high-quality solutions with less than 10% compromise. Since larger ε leads to earlier stopping for the sampling phase, the running time of our algorithm decreases accordingly. Figure 8c confirms a linear decline in approximation ratios with ε we can see that the approximation ratio decreases linearly with the increase of ε as ε directly scales the data-dependent approximation ratio. The results trade-off underscore JB-PIUS 's ability to balance efficiency and precision through tunable ε.



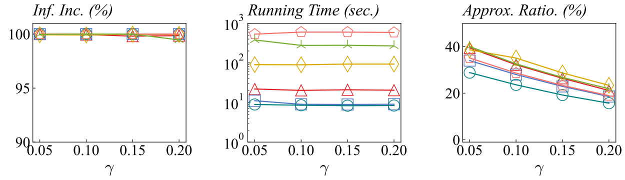
(a) *Varying* $|S|$ (b) *Varying* μ

Figure 7: Running time for varying $|S|$ and μ.



(a) *Influence increment* (b) *Running time* (c) *Approx. ratio*

Figure 8: Varying ε.

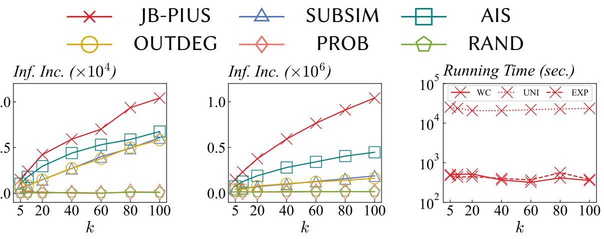


(a) *Influence increment* (b) *Running time* (c) *Approx. ratio*

Figure 9: Varying γ.

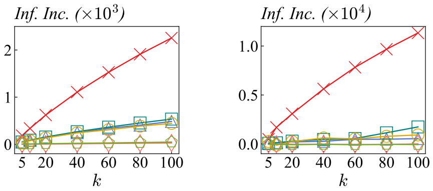
Effect of γ. Now we evaluate our method's sensitivity to γ, the results of which are given in Figure 9. Since the parameter γ solely governs the final selection among A_L, A_U and A_H, the resulting influence increment exhibits minimal variation with increasing γ. As for efficiency, runtime improvements remain marginal across most datasets, as the estimation phase contributes negligibly to total execution time. The approximation ratio declines due to the $\frac{1+\gamma}{1-\gamma}$ term, which directly scales the data-dependent ratio.

Varying probability distribution. To evaluate our algorithm's robustness across probability distributions beyond the default (WC), we also test two additional settings on a fixed seed set and candidate edges - (i) uniform (UNI) [21, 27, 43, 56]: the propagation probability of each edge is set to the same value 0.001, and (ii) exponential distribution (EXP) [21, 44, 56]: the propagation probability of each edge is randomly generated according to the probability density function $f(x) = \lambda e^{-\lambda x}$, where λ is set to 1, and for each node v, the sum of the probabilities of its incoming edges is scaled to 1. The results are shown in Figure 10. It is obvious that JB-PIUS generates higher influence increment to other baselines, indicating JB-PIUS's



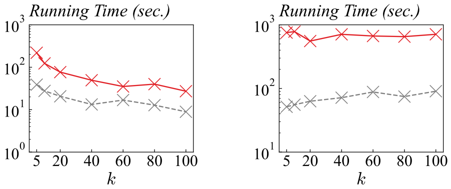
(a) ***Inf. Inc. (UNI)*** (b) ***Inf. Inc. (EXP)*** (c) ***Running time***

Figure 10: Performance on Twitter with different probability distributions.



(a) ***DBLP*** (b) ***Orkut***

Figure 11: Influence increment on undirected datasets with random orientation.



(a) ***DBLP*** (b) ***Orkut***

Figure 12: Running time on DBLP and Orkut with random orientation (the gray dashed lines are the results under the bi-directional setting).

robustness against different influence probability distributions. Notably, the influence increment under the uniform cascade model is smaller than that under other two models by about two orders of magnitude. Correspondingly, the running time for UNI is higher than WC and EXP by over one order of magnitude.

Effect of the orientation on undirected graphs. For the undirected datasets DBLP and Orkut, we test our algorithm's performance when using the random orientation for each undirected edge. Figure 11 shows that the influence increment of JB-PIUS is at least three times higher than the baselines on both datasets. Compared to Figure 3, the increment is smaller with random orientation, as the graph is less connected, lowering the influence. Hence, as shown in Figure 12, the running time with the random orientation is longer, conforming to the complexity analysis in Theorem 5.9.

7 ADDITIONAL RELATED WORK

Influence maximization. Domingos and Richardson [15] are pioneers to take advantage of the word-of-mouth effect to enhance purchasing behavior within social networks. Then, Kempe et al. [27] formalize the well-known influence maximization (IM) problem. They show that this problem is NP-hard, and propose a simulation-based greedy algorithm to approximate the solution. Subsequently, a plethora of research works study how to improve the efficiency to tackle the problem [3, 6–8, 18, 19, 26, 32, 37], most of which are heuristic and fail to provide approximation guarantees. Later, Borgs et al. [4] achieve a notable advancement by introducing an algorithm that runs in near-linear time under the IC model, maintaining a $(1-1/e-\varepsilon)$-approximation ratio. They present the concept of reverse influence sampling (RIS), which underpins the development of state-of-the-art solutions for IM [21, 24, 36, 41, 44, 45]. Different from the discrete scenario, Yang et al. [52, 53] study continuous IM, which aims to offer different discounts to customers with a limited budget such that the resulting influence is maximized.

Topology optimization for IM. In addition to the focus on identifying influential nodes, the subject of enhancing network connectivity to amplify influence propagation has also garnered significant interest from researchers [5, 12, 14, 25, 28, 29, 50, 51]. This line of research aims to maximize the influence spread of a predetermined seed set by inserting a limited number of edges into the social network. Although the objectives are roughly similar, the settings of existing work are various. To name a few, Chaoji et al. [5] study the problem of adding k edges for each node to maximize the influence under a proxy model for IC. Khalil et al. [28] consider adding edges to maximize influence spread under the LT model, but the objective function is the sum of influence spread of each seed. The IMA problem proposed by D'Angelo et al. [14] aims to select k edges incident to the seed set to maximize the influence of the seed set S. The objective function is monotone and submodular [14], and several greedy-based algorithms [9, 12] have been proposed to achieve an approximation ratio of $(1-1/e-\varepsilon)$ [35] for IMA. All of the aforementioned work assumes deterministic seeds, which presents limited generalizability for uncertain seeding scenarios.

8 CONCLUSION

In this paper, we study the problem of inserting links to promote the influence of uncertain seeds in social networks. We first formulate it as an NP-hard combinatorial optimization problem and prove that the objective function is monotone but neither submodular nor supermodular. To provide a solution with certain theoretical guarantees, we instantiate the sandwich approximation strategy with two newly proposed monotone and submodular bounding functions. For the detailed implementation, we propose a joint baking algorithm named JB-PIUS consisting of two efficient estimators for the bounding functions as well as some well-designed greedy selection algorithms. Finally, we conduct extensive experiments to validate our method's effectiveness and efficiency.

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