Maximum k-Plex Finding: Choices of Pruning Techniques Matter!

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ABSTRACT

A k-plex is a dense subgraph structure where every vertex can be disconnected with at most k vertices. Finding a maximum k-plex (MkP) in a big graph is a key primitive in many real applications such as community detection and biological network analysis. A lot of MkP algorithms have been actively proposed in recent years in top AI and DB conferences, featuring a broad range of sophisticated pruning techniques. In this paper, we study the various pruning techniques from nine recent MkP algorithms including kPlexT, Maple, Seesaw, DiseMKP, kPlexS, KpLeX, Maplex, BnB and BS by unifying them in a common framework called U-MkP. We summarize their proposed techniques into three categories, those for (1) branching, (2) upper bounding, and (3) reduction during subgraph exploration. We find that different pruning techniques can have drastically different performance impacts, but there exists a configuration of the techniques dependent on k that leads to the best performance in vast majority of the time. Interestingly, extensive experiments with our unified framework reveal that some techniques are not effective as claimed in the original works, and we also discover an unmentioned technique that is actually the major performance booster when $k > 5$. We also study problem variants such as finding all the MkPs and finding the densest MkP (i.e., with the most edges) to cover community diversity, and effective algorithm parallelization. Our source code is released at https://github.com/akhlaqueak/MKP-Study.

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The source code, data, and/or other artifacts have been made available at https://github.com/akhlaqueak/MKP-Study.

1 INTRODUCTION

Finding cohesive subgraphs in a large graph is useful in various applications, such as finding protein complexes or biologically relevant functional groups [28, 33, 43, 52] and social communities [42, 48].

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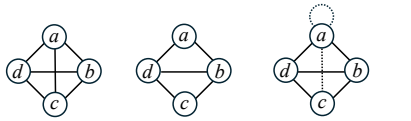


(a) A Clique (b) A 2-Plex (c) Missing Edges of a

Figure 1: Examples of a Clique and a 2-Plex

One classic notion of cohesive subgraph is *clique* which requires every pair of distinct vertices to be connected by an edge (see Figure 1(a)). However, in real graphs, communities rarely appear in the form of cliques due to various reasons such as the existence of data noise [37, 38, 49, 54]. As a relaxed clique model, k-plex was first introduced in [51], which is a graph where every vertex v is adjacent to all but at most k vertices (including v itself). Figure 1(b) shows a 2-plex where every vertex is not adjacent to at most $k = 2$ vertices. For example, a is not connected to $\{a, c\}$ (see dotted edges in Figure 1(c)). It has found extensive applications in the analysis of social networks [51], especially in the community detection [38, 49]. However, mining k-plex structures is NP-hard [29, 47], so existing algorithms rely on branch-and-bound search which runs in exponential time, but utilize effective pruning techniques to make the search process tractable on medium-sized graphs.

Let us focus on the problem of finding a maximum k-plex (MkP), i.e., when there are ties, an algorithm only needs to return one of the maximum k-plexes (other variants will be studied in Section 6). Surprisingly, in recent years, there is a surge of algorithms with new pruning techniques proposed by the AI and DB communities to significantly speed up MkP computation. However, they share a lot of common pruning techniques, many of which are just reinventing the wheels. Moreover, some new techniques may not improve performance at all, or only improve performance for small k values but can lead to catastrophic performance when k is large. However, these issues were not explicitly reported by the respective papers. Even worse, there is a work [53] whose implementation is totally different from what was proposed in its paper, and its performance gain mainly comes from an unmentioned technique. Without a thorough experimental study of these techniques under a unified framework, claims in some of these papers can mislead users to adopt the pruning techniques that result in performance pitfalls.

This paper provides a timely (and in-depth) summary and experimental study of the various techniques proposed by the recent MkP algorithms, by placing them into a unified algorithmic framework. We categorize the pruning techniques applied during subgraph exploration into three categories, those for (1) branching, (2) upper bounding, and (3) reduction. To be self-contained, we provide the

proofs of all techniques in our full technical report [27] with intuitive diagrams and consistent notations, so that the main paper can focus on providing intuitions about the idea behind these techniques. The goal is to provide a benchmark of MkP with which future works can avoid reinventing the wheel and focus on what are really new, and to serve as a comprehensive testbed of the pruning techniques on their performance impacts. We also provide algorithms for variants of MkP that find all the MkPs or the MkP with the most edges to cover community diversity, and parallel versions of all our algorithms.

The insightful experimental findings are summarized as follows:

(1) Different pruning techniques can have drastically different performance impacts (e.g., by thousands of times), but there exists a configuration of the techniques (dependent on k only) that leads to the best performance in vast majority of the time.
(2) The AI community actively designs the upper-bound-based techniques to prune an entire branch of unpromising subgraph search space, but those techniques are only useful for branching instead [44] when k is small. Moreover, a sophisticated strategy such as the one by Seesaw [62] is not more beneficial than a simple one. Even worse, branching in this way can backfire when k is large (not tested and reported in [44]).
(3) When k is large, an unmentioned pivot-based branching method works the best (up to thousands of times faster than upper-bound-based branching), which we extract from the code of Maple [53]. In fact, we find that [53] describes a totally different algorithm which is not actually implemented.
(4) The DB community focuses on designing reduction rules to reduce the size of candidate sets for subgraph expansion during exploration. The latest algorithm kPlexT [35] proposes a new branching method to improve worst-case time complexity, but empirically we find it not competitive to those proposed by the AI community. In contrast, kPlexT manages to find a new reduction rule that continues to significantly improve the search performance. Its prior version kPlexS [34] advocates an incremental reduction technique called CTCP; but we find that CTCP is only worthwhile at the top-level subgraph exploration, but it backfires if it is further applied with the lower-level branches due to the incurred overheads.

The main contributions of this paper are summarized as follows:

(1) To enable the discovery of the 4 findings above, we summarize the pruning techniques of nine state-of-the-art MkP algorithms into three categories, and place them into a carefully-designed unified algorithmic framework called Unified MkP (abbr. U-MkP) to facilitate the flexible configuration of techniques. We utilize U-MkP for extensive experimental studies.
(2) Through the experimental studies, we obtain the above 4 findings which clearly show what techniques work and what do not, which the existing papers fail to reveal. We also identify a configuration of the techniques (dependent on k only) that leads to the best performance in vast majority of the time, and recommend concrete alternatives to try in the rare cases when this configuration is slow (which are difficulty to forecast).
(3) We formalize an unmentioned pivot-based branching method that is the key to the performance of MkP when $k > 5$. We also generalize [44]'s partition-based branching method (using S-based upper bounding) to work with R- and SR-based upper bounding, and it supports additional branch pruning.
(4) To ensure efficiency, we design efficient container structures such as dual-array and auxiliary buffers that are preallocated and incrementally reused/updated during the recursive subgraph exploration to maintain the necessary vertex sets.
(5) We provide algorithms for variants of MkP that find all MkPs or the MkP with the most edges, to avoid missing important dense communities due to returning only one MkP.
(6) We parallelize U-MkP using a task-based approach with time-out mechanism for load balancing to scale up almost ideally.

In the sequel, Section 2 introduces our notations and the branch-and-bound framework adopted by MkP algorithms for subgraph exploration. Then, Section 3 overviews our U-MkP framework and introduces the types of pruning techniques focusing on upper bounding ones. Subsequently, Section 4 summarizes the various branching methods, and Section 5 summarizes the various reduction methods. We discuss the MkP variants and parallelization in Section 6. Finally, Section 7 reports our comprehensive experiments, Section 8 reviews the related works, and Section 9 concludes this paper.

2 PRELIMINARIES

Notations. We consider an undirected and unweighted simple graph $G = (V, E)$, where V is the vertex set, and E is edge set. The degree of a vertex v is denoted by $d_G(v) = |N_G(v)|$. We also define the concept of *non-neighbor*: a vertex u is a non-neighbor of v in G if $(u, v) \notin E$. Accordingly, the set of non-neighbors of v is denoted by $\overline{N_G}(v) = V - N_G(v)$, and we denote its cardinality by $\overline{d_G}(v) = |\overline{N_G}(v)|$. Given a vertex subset $S \subseteq V$, we denote by $G[S] = (S, E[S])$ the subgraph of G induced by S, where $E[S] = \{(u, v) \in E \mid u, v \in S\}$. We simplify the notation $N_{G[S]}(v)$ to $N_S(v)$, and define other notations such as $N_S(v)$, $\overline{N_S}(v)$ and $d_S(v)$ in a similar manner. For an arbitrary graph g, $V(g)$ and $E(g)$ denote the vertex set and edge set of g, respectively. The diameter of G, denoted by $\Delta(G)$ is the shortest-path distance of the farthest pair of vertices in G, measured by the number of hops.

Problem Definition. We next define the concept of k-plex and MkP.

Definition 2.1. (*k-Plex*) A graph g is a k-plex if every vertex $v \in V(g)$ has at least $|V(g)|-k$ neighbors in g, i.e., $d_g(v) \geq |V(g)|-k$. Equivalently, g is a k-plex if every vertex $v \in V(g)$ has at most k non-neighbors in g (including v itself as a non-neighbor), i.e., $\overline{d_g}(v) \leq k$.

Definition 2.2. (*Maximum k-Plex Finding*) Given a graph G, the maximum k-plex finding problem finds a largest vertex set $P \subseteq V$ such that the subgraph $G[P]$ induced by P is a k-plex.

The above MkP finding problem only finds one of the potentially many maximum k-plexes (i.e., MkPs) in G, but all of them could be interesting since they may correspond to different (and even non-overlapping) communities in a social network. Moreover, even all MkPs are ties in terms of vertex number, some may have more edges than others and it would be interesting to find the densest one among them. We, therefore, also consider two problem variants below:

Definition 2.3. (*Finding All MkPs*) Given a graph G, the problem finds all largest vertex sets $P \subseteq V$ such that $G[P]$ is a k-plex.

Definition 2.4. (*Finding the Densest MkP*) Given a graph G, the problem finds an MkP in G with the largest number of edges.

In Appendix A of our technical report [27], we show a case study where it is necessary to mine multiple MkPs to cover different

Algorithm 1: Basic Branch-and-Bound Search

```
1 function BB_basic(S, R, g)    # The basic BB(.) variant
2     if (reduce_and_prune(S, R, g) = true) then return
3     for each v ∈ R do
4         BB_basic(S ∪ {v}, R − {v}, g)
5         R ← R − {v}
```

important communities. Note that the algorithms for these problem variants are just variants of our MkP algorithm (see Section 6).

Hereditariness and Diameter of k-Plex. Note that k-plex satisfies the hereditary property which says that: any induced subgraph (denoted by g') of a k-plex g is also a k-plex, since a vertex in g' cannot miss more neighbors than those already missed in g.

THEOREM 2.5. (Hereditariness) *Given a k-plex $P \subseteq V$, any subset $P' \subseteq P$ is also a k-plex.*

The proof is in Appendix C of our technical report [27].

Moreover, as proved by [55], the diameter of k-plexes with a reasonably large size (which is usually the case for MkPs) is bounded:

THEOREM 2.6. *For a k-plex P and any integer $c \geq 2$, if $|P| > 2k - c$, then $\Delta(P) \leq c$ [55].*

Most works [34, 53] only consider the case when $c = 2$ by assuming $|P| \geq 2k - 1$, which gives the following corollary:

THEOREM 2.7. *Given a k-plex P, if $|P| \geq 2k - 1$, then $\Delta(P) \leq 2$.*

We provide the proof of this special case in Appendix D [27].

The assumption $|P| \geq 2k - 1$ is reasonable for a maximum k-plex P, since natural communities that our MkP problems aim to discover are connected, and $2k - 1$ is relatively small. For example, when $k = 5$, we only require $|P| \geq 9$. Note that a k-plex with $|P| = 2k - 2$ may be disconnected (e.g., formed by two disjoint $(k - 1)$-cliques).

Subgraph Exploration by Branch and Bound. All existing MkP algorithms follow (but are variants of) the branch-and-bound search framework shown in Algorithm 1, where S is the set of vertices already included into the current subgraph, and R is the set of candidate vertices yet to be added to S to form a larger subgraph that can become an MkP. Specifically, given a vertex-set pair $\langle S, R\rangle$ in graph g, Line 2 first calls a function reduce_and_prune(S, R, g) to apply reduction and upper-bounding rules, which may decide that the entire search branch to extend S is unpromising so that *true* is returned, in which case Line 2 returns directly to terminate the extension of S. We will discuss this function in more detail when we introduce Algorithm 3 in Section 3. If the branch is not pruned, Line 3 then takes the next candidate $v \in R$, and split the search space into two cases: (1) v is in the MkP to find, in which case Line 4 further extends $\langle S \cup \{v\}, R - \{v\}\rangle$ by recursion; (2) v is not in the MkP to find, in which case Line 5 removes v so that it will not appear in future iterations of the for-loop in Line 3.

Figure 2 illustrates the search process of Algorithm 1 on a toy graph with four vertices a, b, c and d, which corresponds to a set-enumeration search tree where each node denotes S and we also annotate its corresponding R near the node (assuming that no node is pruned, and that vertices in R are always ordered with $a < b < c < d$).

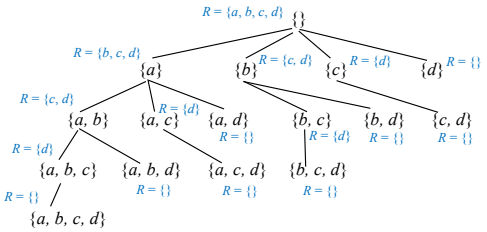


Figure 2: Set-Enumeration Search Tree

We can see that the search space is perfectly partitioned without redundancy. Note that calling Algorithm 1 on S basically grows the set-enumeration subtree under node S, which we denote by T_S.

Also note that we do not need to create new input sets $\langle S \cup \{v\}, R - \{v\}\rangle$ at Line 4, but can reuse the space of $\langle S, R\rangle$, and operations like removing v from R can be done in $O(1)$ time, using the dual-array data structure introduced in Appendix I [27] with Figure 15.

Degeneracy Ordering. The k-core of a graph G is its largest induced subgraph with minimum (vertex) degree k. The degeneracy of G, denoted by $D(G)$, is the largest value of k for which a k-core exists in G. It is well-known that the degeneracy of a graph can be computed in linear time by a peeling algorithm that repeatedly removes a vertex with the minimum current degree at a time [30], which produces a degeneracy ordering of vertices. Appendix B of our technical report [27] provides an illustration of this process. Note that $D(G)$ is usually a small value (see Table 2) since while a high-degree vertex tends to appear later in the degeneracy ordering, so many of its neighbors could have already been removed by peeling.

3 U-MkP: A UNIFIED MkP FRAMEWORK

Initialization. Before subgraph exploration, we first aim to find a large (though may not be maximum) k-plex P as well as an upper bound ub on the size of any MkP in G, so that (1) if $|P| = ub$, then P is already an MkP and can be directly returned, otherwise (2) we can still prune those search branches that cannot lead to a larger k-plex with size at least $(|P|+1)$. We follow kPlexS [34] to compute $\langle P, ub\rangle$ while running the peeling algorithm in linear time, and Appendix E provides the detailed algorithm and its explanations.

Top-Level Branching. Recall from Theorem 2.6 that reasonably large k-plexes ($|P| > 2k - c$) have diameter $\Delta(P) \leq c$. In other words, for the top-level nodes v_i ($v_i = a, b, c, d$) in Figure 2, the branch $T_{\{v_i\}}$ under it only needs to consider a subgraph of G, denoted by g_i, where vertices are within c hops from v_i. It is thus worthwhile to shrink G to g_i, and explore subgraphs in the branch $T_{\{v_i\}}$ over g_i since (1) g_i is much smaller than G so checking the neighbors of each vertex becomes much faster, and (2) this is an efficient one-time processing that can benefit the entire branch $T_{\{v_i\}}$. So, we follow this approach which is also adopted by recent algorithms such as kPlexT [35], kPlexS [35], and Maple [53]. In contrast, when processing $T_{\{v_i\}}$, even though whenever a vertex v is added to S, we can shrink g_i further to remove vertices $> c$ hops away from v, this cost is associated with each expansion of S and so not worthwhile.

Algorithm 2 shows the pseudocode to create top-level search branches $T_{\{v_i\}}$, where we assume there are two globally maintained variables accessible any time during subgraph exploration: (1) P: the current largest k-plex found, and (2) ub: size upper bound of an MkP. These two variables are initialized in Line 2 using Algorithm 6 in

Algorithm 2: Top-Level Branching ($c = 2$)

Input: A graph $G = (V, E)$ and an integer $k \geq 2$
Output: A maximum k-plex in G

1 **function** top_branching(G)
2 $P, ub \leftarrow$ find_init(G) # global variables set by Algo. 6
3 **if** $|P| \geq ub$ **then return** P
4 CTCP(G, $\emptyset$, $|P| + 1 - k$, $|P| + 1 - 2k$)
5 Sort $[v_1, \ldots, v_n]$ following degeneracy ordering of G
6 **for** $i = 1$ *to* $|V|$ **do**
7 $H_1 \leftarrow N(v_i) \cap \{v_{i+1}, \ldots, v_n\}$
8 $H_2 \leftarrow N(H_1) \cap \{v_{i+1}, \ldots, v_n\}$
9 $g \leftarrow$ the subgraph of G induced by $(\{v_i\} \cup H_1 \cup H_2)$
10 BB($\{v_i\}, V(g) - \{v_i\}, g$) # call a BB variant
11 Remove v_i from G
12 CTCP(G, $\{v_i\}$, $|P| + 1 - k$, $|P| + 1 - 2k$)
13 **return** P

Appendix E, and during subgraph exploration, ub stays the same while P will be updated whenever a larger k-plex is found, so that if we know that a branch cannot generate a k-plex larger than $|P|$, it can be pruned without exploration. If $|P|$ equals ub, then the initial P is already a MkP and is thus returned in Line 3. Otherwise, Line 4 then further prunes unpromising vertices and edges from G using the CTCP reduction technique from kPlexS [34] which we will introduce soon, so that top-level branches are created from the pruned G.

We then create branches $T_{\{v_i\}}$ in the degeneracy ordering as shown in Lines 5-6, which keeps the sizes of all g_i's small to avoid having a giant branch that needs to search a deep set-enumeration subtree with potentially high node fanouts. This is because a high-degree vertex v_i tends to appear later in the degeneracy ordering, so V_i has excluded many neighbors of v_i originally in V.

Without loss of generality, let us assume that $c = 2$, that is, an MkP P has $|P| \geq 2k-1$ (i.e., $|P| > 2k-2$). To create g_i for each v_i, Line 7 first obtains one-hop neighbors of v_i in V_i, and Line 8 then uses them to obtain the two-hop neighbors (where $N(H_1) = \cup_{v \in N(H_1)} N(v)$); finally, Line 9 creates g_i using them (i.e., excluding other vertices in V_i that are > 2 hops away from v_i). Note that it is easy to extend g_i for the general case of c. For example, when $c = 3$, we can additionally compute $H_3 \leftarrow N(H_2) \cap \{v_{i+1}, \ldots, v_n\}$ and then compute g_i to be the subgraph of G induced by $(\{v_i\} \cup H_1 \cup H_2 \cup H_3)$.

Once g_i is created, Line 10 then explores it by extending $S = \{v_i\}$ using branch and bound. For now, we can think of BB(.) simply as BB_basic(.) presented in Algorithm 1, and Section 4 will describe two variants using more efficient branching methods. We then remove v_i from G in Line 11 (similar to Line 5 of Algorithm 1) so that later branches will not consider v_i to avoid redundancy. Note that we do not actually need to do the intersections in Lines 7 and 8 since $\{v_1, \ldots, v_{i-1}\}$ have already been removed in previous iterations.

Core-Truss Co-Pruning (CTCP). kPlexS [34] proposes a CTCP technique to prune the vertices and edges using the fact that to find a larger k-plex with size at least $|P|+1$, we only need to consider those vertices with degree at least $\tau_v = (|P| + 1 - k)$ and those edges (u, v) where u and v share at least $\tau_e = (|P| + 1 - 2k)$ common neighbors (see Theorem D.1 in our technical report [27]). Figure 3 shows an illustration where vertices and edges below thresholds are alternately

Algorithm 3: Reduction and Pruning Function

1 **function** reduce_and_prune(S, R, g) # returns if T_S is pruned
2 **if** ($|P| = ub$) **then return** *true*
3 Apply reduction rules to (S, R, g)
4 **if** $g_{union} = g[S \cup R]$ *is a k-plex* **then**
5 **if** $|S| + |R| > |P|$ **then** $P \leftarrow S \cup R$
6 **return** *true*
7 **if** partition(S, R, g) $= \emptyset$ **then return** *true* # Seesaw UB
8 **return** *false*

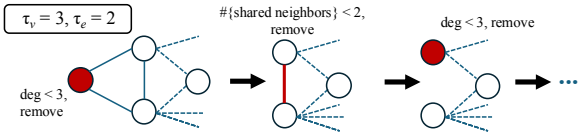


Figure 3: Illustration of CTCP

pruned. kPlexS [34] shows that the reduced graph by CTCP is guaranteed to be no larger than that computed by BnB [40], Maplex [63] and KpLeX [63] and was then the most efficient MkP algorithm.

We adopt the efficient implementation of CTCP(G, Q_v, *lb_changed*, τ_v, τ_e) from kPlexS [34], where Q_v keeps the set of vertices that must be removed from G, and *lb_changed* is a flag indicating if the current largest k-plex P has changed (so that τ_e is increased and more edges may be pruned). By maintaining all $d_G(v)$ for all v and the triangle counts for all edges (u, v) (denoted by $d_G(u, v)$), and dynamically updates them while propagating pruning from vertices in Q_v (and if *lb_changed* = *true*, also from new edges with $d_G(u, v) < \tau_e$), CTCP can minimize its graph update footprint and overhead to shrink G.

Refer back to Algorithm 2 where for simplicity, we omit argument *lb_changed* of CTCP. Line 4 conducts CTCP over the entire G before creating branches (with *lb_changed* = *false*). Moreover, after Line 11 removes v_i from G, new edges (e.g., v_i's adjacent ones) may have $d_G(u, v)$ reduced below τ_e, so CTCP is called in Line 12 with $Q_v = \{v_i\}$ to shrink G further for use by future iterations (Here, *lb_changed* is determined by whether BB(.) called in Line 10 has found a larger k-plex to update P).

Reduction and Upper-Bound-Based Branch Pruning. Recall from Line 2 of Algorithm 1 that BB(.) first calls reduce_and_prune(.) over the instance (S, R, g) for further reduction before extending S with vertices from R, the pseudocode of which is shown in Algorithm 3. Specifically, Line 1 first checks if the current P already reaches the maximum possible size ub, and if so, it returns *true* to notify BB(.) to terminate its subgraph exploration (see Line 2 of Algorithm 1). Right after the call of BB(.) in Line 10 of Algorithm 2, we can check if $|P| = ub$ and return P directly if so to terminate early.

Otherwise, we apply reduction rules (to be introduced in Section 5) in Line 3 to reduce the candidate size of R. Before extending S with vertices in R (e.g., Lines 3–5 in Algorithm 1), we first conduct a look-ahead pruning in Line 4 to see if the entire $g_{\text{union}} = g[S \cup R]$ is a k-plex, and if so, Line 6 returns true so that Line 2 of Algorithm 1 will return directly to terminate its exploration of T_S. Moreover, if $S \cup R$ is larger than the current P, P is updated with $S \cup R$ in Line 5.

Finally, we check if T_S can still produce a k-plex larger than P, and if not, Line 7 returns *true* to let BB(.) terminate its exploration of T_S. The function partition(.) (to be introduced in Section 4) uses

Table 1: MkP Algorithms and Their Pruning Techniques

Algorithm	Upper Bounding (UB)					Branching					Reduction						
	S-Based	R-Based	SR-Based	UBR1	UBR2	Binary	Pivot	S-Based	R-Based	SR-Based	Two-Hop	RR1	RR2	RR3	CTCP	BR1	BR2
U-MkP (Our Work)	✓	✓	✓	✓			✓	✓	✓	✓	✓^{g}	✓	✓	✓	✓^{T}	✓	✓
kPlexT (SIGMOD'24)				✓		✓					✓^{g}	✓	✓	✓	✓^{T}	✓	✓
Maple (IJCAI'23)				✓			✓				✓^{g}	✓	✓	✓	✓$^{+}$	✓	✓
Seesaw (arXiv'23)	✓$^{+}$	✓$^{+}$	✓	—	—	—	—	—	—	—	—	—	—	—	—	—	—
DiseMKP (IJCAI'23)	✓$^{+}$							✓$^{+}$				✓	✓	✓	✓^{T}		
kPlexS (PVLDB'22)				✓		✓					✓^{g}	✓	✓	✓	✓$^{+}$	✓	✓
KpLeX (IJCAI'21)	✓							✓				✓	✓	✓			
Maplex (AAAI'21)		✓				✓					✓	✓	✓	✓	✓^{T}		
BnB (IJCAI'18)						✓						✓	✓	✓			
BS (AAAI'17)							✓				✓	✓	✓	✓			✓

*** Note:** ✓^{g}: Two-hop graph constructed; ✓$^{+}$: Improved version; ✓^{T}: Top-level only
—: Seesaw only proposes UB rules, no open-source code;

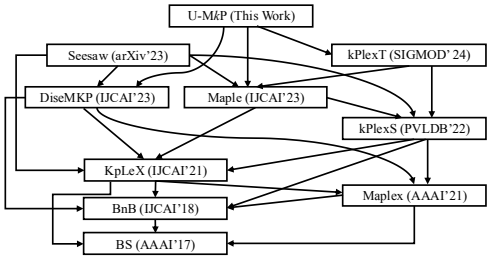


Figure 4: Reported Algorithm Dominance Relationships

upper-bound-based pruning to return the subset of vertices in R, denoted by B, that is still worth branching with (i.e., to extend S only with a vertex in B in the next step), and if $B = \emptyset$, there is nothing worth extending with so the entire T_S is pruned.

Overview of Pruning Techniques. We now place the various pruning techniques of nine recent algorithms under our U-MkP framework established in Section 3, including kPlexT [35] Maple [53], Seesaw [62], DiseMKP [44], kPlexS [34], KpLeX [45], Maplex [63], BnB [40] and BS [55]. Table 1 summarizes the pruning techniques of these algorithms into three categories, those for upper bounding, branching, and reduction. We can see that most algorithms are from the AI community (AAAI and IJCAI), since we can regard MkP finding as a search problem. Specifically, the set-enumeration tree defines a state space where each node S is a state, extension set R defines the successor function, and an MkP for S to expand into is a goal state. Moreover, the two works kPlexT [35] and kPlexS [34] are from the DB community, actually the same group. Figure 4 shows the performance dominance relationships reported by the respective papers to their compared algorithms.

We have covered two reduction rules "Two-Hop" (or more generally, "c-Hop") and "CTCP" since they are essential for describing our U-MkP framework, and we will introduce the other reduction rules in Section 5. Branching rules will be introduced in Section 4.

Regarding upper-bounding rules, one usage is to prune an entire unpromising branch T_S as mentioned in Line 7 of Algorithm 3, but since it is also applied by a branching strategy, we will present the details in Section 4.1 next. Note that there are still two more upper-bounding rules UBR1 and UBR2 in Table 1, and since they are used for reduction, we will present them in Section 5.

Also, Table 1 shows that BS uses more techniques than BnB but is slower. This is because instead of finding a large initial k-Plex as in Algorithm 6 to enable effective pruning (e.g., by CTCP in Line 4 of Algorithm 2), BS solves a decision version of MkP that guesses its size, and relies on binary search on the size range to locate MkP by running explorations for up to $O(\log |V|)$ times [55].

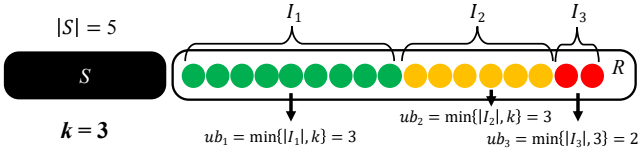


Figure 5: Illustration of Partition-Based Upper Bounding

4 BRANCHING METHODS

This section reviews two competitive branching methods, one using **partition-based upper bounding**, and the other using **pivoting**, both significantly beating a baseline binary branching method. For the former, we show that although the latest cost-model-driven Seesaw upper bound [62] was originally proposed only for entire-branch pruning (see Line 7 of Algorithm 3), it can actually be adapted for branching. For the latter, we actually reverse engineered this unmentioned technique from the code of Maple [53], which unfortunately, implements something totally different from what their paper described. Recall our contribution (3) in Section 1.

4.1 Partition-Based Upper Bounding

The partitioning-based upper bounding technique was proposed and improved by series of works: Maplex [63], KpLeX [45], DiseMKP [44], Seesaw [62]. Given $\langle S, R\rangle$, the goal is to compute a size upper bound UB_S on the largest k-plex that S can be extended into using candidates from R (i.e., within branch T_S), by partitioning vertices of R into t subsets $I_1, I_2, \ldots, I_t$. If for each I_i, we can show that at most ub_i vertices from I_i can be added to S without breaching the k-plex requirement, then UB_S can be computed as $|S| + \sum_{i=1}^{t} ub_i$.

We illustrate by the example in Figure 5 that uses Maplex's color-based method, which repeatedly removes a maximal independent set I_i from R at a time. Since vertices in I_i do not have any mutual edges (so can share the same color during graph coloring, hence the name), if more than k vertices are added from I_i to S, then each added vertex will have more than k non-neighbors in S, breaching the k-plex requirement and the set cannot be further extended into a k-plex due to the hereditary property. Therefore, $ub_i = \min\{|I_i|, k\}$. In Figure 5, the upper bound is given by $UB_S = |S| + \sum_{i=1}^{3} ub_i = 5+3+3+2 = 13$.

R-Based Upper Bounding. We present the above algorithm to obtain each $\langle I_i, ub_i\rangle$ in Appendix F of our technical report [27]. We call this method as **R-based strategy**, since it computes independent sets directly from R without referring to S's content.

Seesaw [62] further tightens this bound. To explain its method, we first define the concept of vertex support.

Definition 4.1. (*Support*) The support of a vertex v is defined as $\delta_S(v) = k - |\overline{N_S}(v)|$, which indicates the maximum number of non-neighbors of v that can be added to S.

Intuitively, $\delta_S(v)$ serves as the non-neighbor "quota" of v that can be added to S (including v itself if $v \notin S$) without breaching the k-plex requirement. This support definition is also used a lot in the reduction rules, as we shall see in Section 5.

We actually maintain $d_S(v)$ incrementally and keep it up to date whenever we move vertices around R and S, so $\delta_S(v) = k - (|S| - d_S(v))$ can always be obtained in $O(1)$ time for use by our pruning techniques. See Appendix J [27] for the details.

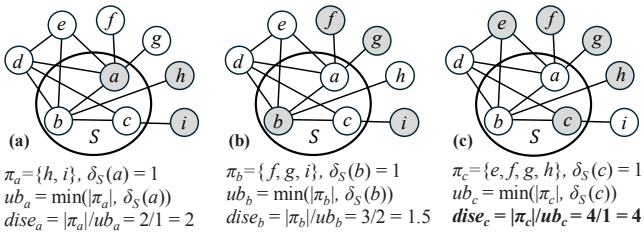


Figure 6: An Example Graph for S-Based Partitioning

Now we are ready to present Seesaw's R-based strategy. Seesaw tightens $ub_i = \min\{|I_i|, k\}$ into $ub_i = \max\{i \mid \delta_S(v_i) \geq i\}$ (see Lemma 1 of [62]). Moreover, for each $\langle I_i, ub_i\rangle$ obtained where I_i is a maximal independent set, Seesaw [62] further relaxes I_i to include additional vertices from R without increasing the value of ub_i (see Lemmas 2–3 of [62]). This reduces the number of remaining vertices in R (from which future partitions are obtained) hence tends to reduce the upper bound UB_S. Our current work uses this improved approach to obtain $\langle I_i, ub_i\rangle$ rather than the simple approach of Maplex, and we denote this operation by

$$\langle I_i, ub_i\rangle \leftarrow \text{get\_R\_part}(R, S, g)$$

where get_R_part(.) is specified by Algorithm 2 of [62].

S-Based Upper Bounding. KpLeX [45] proposes a different way to obtain $\langle I_i, ub_i\rangle$ by partitioning vertices of R based on vertices in S, so we call this approach **S-based strategy**. Specifically, it partitions R by obtaining from R (1) the common neighbors of all vertices in S as π_0, (2) for each $v_i \in S$, obtain π_i as all the remaining non-neighbors of v_i in R. Consider the graph in Figure 6(a) where $S = \{a, b, c\}$ and vertices d to i belong to R: if we check vertices in S by the order $[a, b, c]$, we obtain $\pi_0 = \{d\}$, $\pi_1 = \{h, i\}$, $\pi_2 = \{f, g\}$, $\pi_3 = \{e\}$.

KpLeX treats each π_i $(i \geq 1)$ as I_i, we have $ub_i = \delta_S(v_i)$ since it is the quota of non-neighbors of v_i that can be pulled into S. Clearly, we can compute $UB_S = |S| + |\pi_0| + \sum_{i=1}^{|S|} \min\{|\pi_i|, \delta_S(v_i)\}$. However, the value of UB_S depends on the checking order of vertices in S.

To find an ordering that leads to small UB_S, DiseMKP [44] proposes the concept of distribution efficiency (dise) where $dise(\pi_i) = |\pi_i|/ub_i$. Intuitively, we prefer high $dise(\pi_i)$ since we want to remove a large set π_i from R while adding a small ub_i to UB_S. Based on this idea, DiseMKP proposes to greedily check $dise(\pi_i)$ for all the remaining $v_i \in S$ whose π_i has not been selected, and choose the one with the highest $dise(\pi_i)$ as the next I_i. Figure 6 illustrates how I_1 is determined by computing the dise scores, and π_c is picked as I_1 since it has the highest dise. Note that π_i's are initialized as all non-neighbors of v_i that could overlap, so after $I_1 = \pi_c$ is picked, we need to update $\pi_a = \pi_a - I_1 = \{i\}$ and $\pi_b = \pi_b - I_1 = \{i\}$, and then pick I_2 by comparing their dise scores.

We denote the above operation to obtain a partition I_i as

$$\langle I_i, ub_i\rangle \leftarrow \text{get\_S\_part}(R, S, g, \Pi)$$

where $\Pi = \cup_{v_i \in S} \pi_i$ is an auxiliary set that keeps the (potentially overlapping) non-neighbor sets π_i of all $v_i \in S$, which is initialized before computing UB_S, and tracks the candidate partitions to choose next. Note that whenever a partition π_i is chosen, its vertices are removed from all partitions in Π, so if π is previously chosen, it will become empty. Function get_S_part(.) simply chooses the next partition among those non-empty π_i's in Π, and the detailed algorithm is specified in Appendix G of our technical report [27].

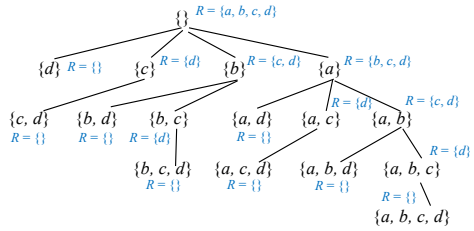


Figure 7: Set-Enumeration Search Tree (Horizontal Flip)

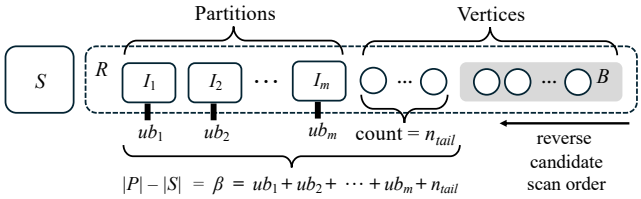


Figure 8: Illustration of Branching Set Computing

SR-Based Upper Bounding. Since $dise(\pi_i) = |\pi_i|/ub_i$ is also well defined for R-based partitions, Seesaw [62] proposes to run both method when picking the next partition from R:

$$\langle I_i^R, ub_i^R\rangle \leftarrow \text{get\_R\_part}(R, S, g),$$
$$\langle I_i^S, ub_i^S\rangle \leftarrow \text{get\_S\_part}(R, S, g, \Pi),$$

and pick the partition with the higher dise as the next partition I_i. Of course, vertices of I_i need to then be removed from R and all candidate sets π_j maintained for the S-based strategy, and ub_i is added to UB_S. This is repeated until R becomes empty (so π_0 will be refined by R-based strategy at the end), after which we add $|S|$ to UB_S to obtain the final UB_S. This method is the **SR-based strategy**.

4.2 Partition-Based Branching

Branching Set. The above partition-based upper bounding technique can actually be used for branching that allows some branches to be pruned, as proposed by DiseMKP [44]. Specifically, let us consider the horizontal flip of the set-enumeration tree in Figure 2, which we show in Figure 7. We can observe that when we scan the candidates from right to left till reaching a candidate $v \in R$, the set-enumeration subtrees rooted at v and all the candidates in front of v in R can only involve vertices in S, v, and those vertices before v in R. For example, for node $S = \{\}$ in Figure 7, when we scan R and reach b, all the three subtrees rooted at nodes $\{d\}$, $\{c\}$ and $\{b\}$ contain vertices in $\{b, c, d\}$. Similarly, for node $S = \{a\}$, when we scan R and reach c, the two subtrees rooted at nodes $\{a, d\}$ and $\{a, c\}$ contain vertices in $\{a, c, d\}$. Now let us consider Figure 8 with $\langle S, R\rangle$, and the last few vertices constitute a set B. Following the above discussion, we know that subtrees rooted at the vertices in $R' = R - B$ can only involve those vertices in $S \cup R'$. If we can show that the size upper bound of a k-plex obtained by extending S with candidates in R', denoted by $UB_S(R')$, cannot exceed $|P|$ (i.e., cannot produce a larger k-plex), then we do not need to extend S with those vertices in R' in the next level. In other words, we only need to branch over vertices in B, hence we call B the branching set.

Note, however, that vertices of R' can still appear in the subtrees of those nodes that extend S with vertices in B, so they are not removed from g (i.e., not reduction) but just skipped for the level of branching below S. To illustrate with Figure 7 again, assume that

Algorithm 4: Partitioning-Based Branch and Bound

Input: Flag σ_{seed}: true if called by the top level.
Flag σ_{up}: a global variable indicating branch pruning

1 {Updates P if a larger maximum k-plex is found}
2 **function** BB_part(S, R, g, σ_{seed})
3 **if** (reduce_and_prune(S, R, g) = *true*) **then return**
4 $B \leftarrow$ partition(S, R, g)
5 $R' \leftarrow R \setminus B$
6 Sort $B = [v_1, \ldots, v_n]$ following degeneracy ordering of g
7 **for** $i = |B|$ *to* 1 **do**
8 **if** ($\sigma_{seed} = true$) **then** $\sigma_{up} \leftarrow false$
9 **if** ($\sigma_{up} = true$) **then return**
10 $\tau_{old} \leftarrow |P|$
11 BB_part($S \cup \{v_i\}, R', g,$ *false*)
12 **if** $|P| > \tau_{old}$ **then** $\sigma_{up} \leftarrow true$
13 $R' \leftarrow R' \cup \{v_i\}$

$R' = \{d, c\}$ for $S = \{\}$, then even if we skip the subtrees under $\{d\}$ and $\{c\}$, c and d can still appear in the subtrees under $\{b\}$ and $\{a\}$.

We can use the partition-based upper bounding techniques presented in Section 4.1 to construct R' (and hence $B = R - R'$). As Figure 8 illustrates, our goal is to find R' as a subset of R that is as large as possible so that $UB_S(R') \leq |P|$. Let us view the computation of $UB_S(R')$ as the following process: initially, $UB_S(R')$ is set as $|S|$, and we then partition R' into partitions $\langle I_i, ub_i \rangle$ and add all ub_i values to $UB_S(R')$ to obtain the final value of $UB_S(R')$.

We can use any of the S-based strategy, R-based strategy, or SR-based strategy presented in Section 4.1 to find $\langle I_i, ub_i \rangle$ one at a time and expand R' with I_i. Since we require $UB_S(R') \leq |P|$, we have the quota $\beta = |P| - |S|$ for the summation of ub_i values when we choose as many partitions I_i into R' as possible. Whenever a partition i_i is chosen, we deduct ub_i from β so that β is always the remaining quota. We denote the above process to compute B ($R' = R - B$) as

$$B \leftarrow \text{partition}(S, R, g),$$

which keeps obtaining $\langle I_i, ub_i \rangle$ as long as $\beta \geq ub_i$ (during S-based partition selection as shown in Figure 6, we also do not consider those candidates π_i with $ub_i > \beta$). Finally, as Figure 6 illustrates, assume that $ub_{m+1} > \beta$ for the next partition I_{m+1} (if exists), we then stop obtaining this and future partitions (with potentially lower dise scores), but rather taking β more vertices from R (if exists) to prune more branches by letting $UB_S(R')$ reach the allowed value $|P|$.

Algorithm 9 in Appendix H of our technical report [27] shows the pseudocode of partition(S, R, g) when we apply the most sophisticated SR-based strategy, and the counterparts for the other two strategies can be similarly derived (but much simpler).

Partition-Based Branch and Bound. Operating on the horizontally flipped set-enumeration tree as shown in Figure 7 allows additional branch pruning as follows: assume that we have processed the top-level branches $T_{\{v_1\}}, T_{\{v_2\}}, \ldots, T_{\{v_{i-1}\}}$ and the current largest k-plex is P, then we can find a k-plex with at most $(|P| + 1)$ vertices in branch $T_{\{v_i\}}$, so as soon as we find such a k-plex, we can skip the rest of $T_{\{v_i\}}$ (where we can find at most ties) and move on to $T_{\{v_{i+1}\}}$.

To see this, consider Figure 7 again, and assume that $T_{\{d\}}$, $T_{\{c\}}$ and $T_{\{b\}}$ have been processed so that P is an MkP found over $\{b, c, d\}$. Then in $T_{\{a\}}$, we show that we cannot we find a k-plex P' of size more than $|P| + 1$ by contradiction: assume that $|P'| > |P| + 1$, then since a is the only additional vertex beyond $\{b, c, d\}$, we must have $a \in P'$ (otherwise, P is not an MkP over $\{b, c, d\}$); by the hereditary property of k-plex, $P' - \{a\} \subseteq \{b, c, d\}$ is also a k-plex but has size more than $|P|$, which contradicts with the fact that P is an MkP over $\{b, c, d\}$. This proof can easily be adapted to the general case.

Based on this idea, Algorithm 4 shows the partition-based branch-and-bound algorithm to be called by Algorithm 2 with $\sigma_{seed} = true$. Here, σ_{seed} indicates if the function is called by the top level, and within each branch under a top-level node, the recursive call passes *false* to σ_{seed} as Line 11 shows. As a result, when computation just enters a top-level branch $T_{\{v_i\}}$, Line 8 will initialize the global flag variable σ_{up} to *false* to indicate that a k-plex of size $(|P| + 1)$ has not been found in $T_{\{v_i\}}$ yet. Line 11 then recursively extends $S \cup \{v_i\}$ for a vertex $v_i \in B$, and Line 12 checks if the recursive call has increased $|P|$ beyond its old value recorded in Line 10. If so, a larger k-plex is found (and must have size $|P| + 1$ based on the previous discussion), so we terminate the exploration of $T_{\{v_i\}}$ by directly returning in Line 9 along the backtracking path all the way to the top level, which will then proceed the exploration to $T_{\{v_{i+1}\}}$.

Note from Lines 6–7 that we only branch on vertices in B, and we check vertices from $v_{|B|}$ to v_1 in the reverse degeneracy ordering, so that dense regions are examined first in hope that a larger P can be identified early to prune a lot of unpromising branches.

Implementing Auxiliary Buffers Π. Recall that the S- and SR-based strategies require an auxiliary set $\Pi = \cup_{v_i \in S} \pi_i$ that keeps the (potentially overlapping) non-neighbor sets π_i of all $v_i \in S$. Creating and initializing it for each time when partition(.) is called in Line 4 of Algorithm 4 is very expensive, so we propose to preallocate buffer space for Π to be reused during recursive subgraph exploration in a space-efficient manner. The details can be found in Appendix K [27].

As our experiments shall show, the partition-based branching method is usually the most efficient branching method for the small values of k (2 to 5) that are the focus of most papers (only Maple [53] tested large values for k), but only adopted by S-based methods KpLeX [45] and DiseMKP [44] with the scheme not clearly explained, so later works such as kPlexT [35] still uses binary branching that is less efficient. By clearly explaining this partition-based scheme and extending it to support all 3 variants of partition-based strategies (S-, R-, and SR-based), we hope to motivate future MkP works to consider this partition-based scheme when k is small.

Also note that instead of evaluating if $UB_S \leq |P|$ in Line 7 of Algorithm 3, we evaluate if $B = \emptyset$ in Line 7, which is equivalent.

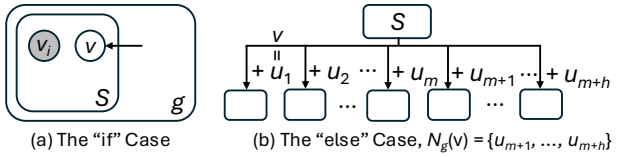
4.3 Binary Branching

As Table 1 shows, most algorithms adopt simple binary branching that given instance $\langle S, R \rangle$, chooses a vertex $v \in R$ with the smallest degree (to follow the degeneracy ordering) and divides into two instances $\langle S \cup \{v\}, R - \{v\} \rangle$ (i.e., P containing v) and $\langle S, R - \{v\} \rangle$ (i.e., P not containing v). However, there is no branch pruned like R' in partition-based method, so simple binary branching is inefficient. Interestingly, kPlexT [35] recently proposes a slightly improved binary branching method to choose v more smartly, which is essential for proving their improved worst-case time complexity. However, our experiments show that their branching method does not obviously

Algorithm 5: Pivoting-Based Branch and Bound

Input: P, B are global variables, v_i is the top-level vertex
$\mathcal{S}$ is the candidate stack
Output: Updates P if a larger maximum k-plex is found

```
1  function BB_pivot(S, R, g)
2      if (reduce_and_prune(S, R, g) = true) then return
3      if S = ∅ or S.top() ∈ S or S.top() ∉ R then
4          if ∃v ∈ R, s.t. (v, v_i) ∉ E(g) and (δ_S(v) = 1
             or δ_S(v_i) = 1 or d_g(v) + k ≤ |P| + 1) then
5              S ← {v}
6          else
7              v_p = A vertex with minimum d_g(.) in R
8              S ← R − N_g(v_p)
9              Sort S in descending order of d_g(.)
10     v ← S.pop()
11     BB_pivot(S ∪ {v}, R − {v}, g)
12     Remove v from g
13     S.clear()
14     BB_pivot(S, R, g)
```



(a) The "if" Case (b) The "else" Case, $N_g(v) = \{u_{m+1}, \ldots, u_{m+h}\}$

Figure 9: Illustration of Pivot Selection

improve performance, and efficiency of kPlexT is mainly contributed by a new reduction rule UBR2 and avoiding CTCP in non-top-level BB recursion (in contrast to their prior algorithm kPlexS).

In contrast, our experiments show that the pivot-based binary branching method of Maple [53], as shown in Algorithm 5, can significantly improve performance, and is often many orders of magnitude faster for branching when k is large. Interestingly, Algorithm 5 is reverse-engineered from Maple's code, and their paper [53] proposes something totally different that finds MkP by solving a complement problem of k-plex called d-BDD, which is not actually implemented in their GitHub repo, so cannot be compared.

Our experiments reveal that while partition-based branching is the most efficient for small k values, it is not competitive to BB_pivot(.) (Algorithm 5) when k becomes larger. This is mainly because each $ub_i = \min\{|\pi_i|, \delta_S(v_i)\}$ becomes loose when k is large (since $\delta_S(v_i) = k - |\overline{N_S}(v_i)|$), reducing the pruning power of BB_part(.) (Algorithm 4). We recommend BB(.) to choose BB_part(.) when $k \leq 5$ and BB_pivot(.) otherwise (see Appendix L [27]).

We now explain Algorithm 5, which treats MkP finding as a constraint satisfaction problem (CSP) and applies the most-constraining-variable heuristic to find candidates to extend S that tend to reduce the remaining candidates in R the most. As we can see from Lines 11, 12 and 14, BB_pivot(.) is still a binary branching method that selects a pivot v at a time to divide the instance. The key success lies in its two methods to select v to be the most constraining.

In Case 1, Line 4 aims to find v as a non-neighbor of top-level vertex v_i (i.e., we are exploring $T_{\{v_i\}}$) which decrements $\delta_S(v_i)$ to promote pruning. For such a v we also require it to fall in one of the three candidate-constraining cases (see Figure 9(a)): **(1)** $\delta_S(v) = 1$, so after v is added to S (hence $\delta_S(v) = 0$), all non-neighbors of v can be pruned from R; **(2)** $\delta_S(v_i) = 1$, so after v is added to S (hence $\delta_S(v_i) = 0$), all non-neighbors of v_i can be pruned from R; **(3)** $d_g(v) + k \leq |P| + 1$, so extending $S \cup \{v\}$ cannot lead to a k-plex larger than $|P| + 1$ (see reduction rule RR3 in Appendix M [27]) and is more likely to be pruned ($d_g(v) + k > |P|$ must hold, or RR3 should have pruned v in Line 2). If v is found, Line 5 will set $\mathcal{S}$ to contain only v, and v will be popped in Line 10 for binary branching.

Otherwise, the else-branch (Case 2) finds $v_p \in R$ as the vertex with minimum $d_g(.)$, and pushes the candidates in R to $\mathcal{S}$ so that they are popped in Line 10 in non-decreasing order of $d_g(.)$ (to follow the degeneracy ordering), with v_p being the first to pop. Note that as Line 8 does, we do not add $N_g(v_p)$ to $\mathcal{S}$ since as Figure 9(b) shows, the last h branches that correspond to candidates from $N_g(v_p)$ can only produce k-plexes $P' \subseteq N_g(v_p)$; since v_p connects to every vertex in P', $P' \cup \{v_p\}$ is also a k-plex, so P' cannot be an MkP.

As we shall see in Section 5, reduction rules in Line 2 may move some candidates directly from R to S, leading to $\mathcal{S}.top() \in S$; and may prune some candidates already added to $\mathcal{S}$, leading to $\mathcal{S}.top() \notin R$, in which case Line 3 will activate the selection of a new pivot. Otherwise, the if-branch in Line 3 will be skipped and next candidate in $\mathcal{S}$ will be popped for binary branching.

5 REDUCTION METHODS

This section briefly overviews the reduction techniques summarized in Table 1, focusing on categorizing them. Due to the space limit, the detailed rules and their proofs can be found in Appendix M [27].

The general idea of reduction rules is to remove unpromising vertices from g. These rules check $d_g(v)$ and $\delta_S(v) = k - (|S| - d_S(v))$ for vertices v frequently, so we incrementally maintain $d_g(v)$ and $d_S(v)$ and keep them up to date whenever we move vertices around R and S, so that $d_g(v)$, $\overline{d_g}(v) = V(g) - d_g(v)$ and $\delta_S(v)$ can be always be obtained in $O(1)$ time for use by our reduction rules below. See Appendix J [27] for the details on incremental degree maintenance.

Section 3 discussed **TwoHop** and **CTCP**. We will now discuss the remaining ones: RR1–RR3, BR1–BR2, and UBR1–UBR2. Note that while kPlexS [34] promotes CTCP for each subgraph extension, we find its overhead to be high, and using the other more efficient reduction rules for reduce_and_prune(.) in BB(.) is more favorable. This is also what was done in their follow-up work kPlexT [35].

RR1–RR3 shrink R. Given (S, R, g), a vertex $v \in R$ is unpromising if $S \cup \{v\}$ is not a k-plex, so **RR1** and **RR2** prune such v from R, based on conditions that check $\delta_S(.)$. Also, v is unpromising if extending $S \cup \{v\}$ cannot produce a k-plex larger than P, so **RR3** prunes such v from R based on a condition that checks $d_g(v)$.

RR1-RR3 are used by all MkP algorithms, so we put them in the same category. There are two more sophisticated upper-bound-based reduction rules that utilize tighter upper bounds (of the size of a k-plex extending $S \cup \{v\}$), so are more powerful in pruning candidates in R: **UBR1** is proposed by kPlexS [34], and **UBR2** by kPlexT [35], both from the same group in the DB community. Notably, our experiments reveal that UBR2 is particularly effective as a reduction rule even though it is only discovered very recently.

Finally, there are also two reduction rules that directly add a vertex $v \in R$ to S, based on conditions related to $d_g(.)$. Specifically, **BR1** is first proposed by kPlexS [34] based on conditions related

to $d_g(v)$ which ensures that, if there exists an MkP P containing S, then there must also exist an MkP P' containing $S \cup \{v\}$. Therefore, if we only need to find one MkP, we can directly move such a v into S, but if our goal is to find all MkPs, we also need to consider the other branch where v is removed from R. In contrast, **BR2** is first proposed by BS [55] based on conditions related to $d_g(.)$ of v and all $u \in \overline{N_g}(v)$ which ensures that, every MkP in g must contain v, so we can safely move such v to S, even when we are finding all MkPs.

6 MkP VARIANTS AND PARALLELIZATION

MkP Variants. For the variant that finds all MkPs (rather than an arbitrary one) which we call as **All-MkP**, we propose a two-phase approach. Phase 1 finds the size of MkPs (denoted by p_{max}) using our MkP framework. Then, Phase 2 loads G again and prunes it using CTCP with $\tau_v = p_{max} - k$ and $\tau_e = p_{max} - 2k$. During subgraph exploration, we lock the threshold of the current largest k-plex size to be $p_{max} - 1$, so that the pruning techniques will not prune the search space that can lead to any k-plex of size p_{max}; and whenever such an MkP is found, it is immediately emitted as a result without incrementing the threshold. Also, we cannot enable BR1 in reduce_and_prune(S, R, g) to avoid missing any MkP.

For finding the MkPs with the most edges (**Densest-MkP**), Phase 2 instead only update the current densest MkP if a newly found one has more number of edges.

All-MkP can find diversified communities that have a tie in size but may not overlap much, while Densest-MkP can further identify a denser community among them. Appendix A [27] provides a case study to show how these variants can avoid missing important dense communities due to returning only one MkP. Note that the algorithms for these problem variants are just variants of our MkP algorithm.

The Parallel Algorithm. Our MkP framework can be easily parallelized with a task-based model [46, 57] that has been extensively applied in compute-intensive graph search and mining problems [36, 41, 46, 56, 58–61]. The idea is to add all $v_i \in V$ into a task queue Q, where each element v_i defines a top-level branch $T_{\{v_i\}}$ as an independent task that can be assigned to an idle CPU core for recursive mining by first creating the two-hop graph (denoted by g_i) created for v_i as shown in Lines 7–9 of Algorithm 2. At any time, we only process a window of θ tasks and hence maintain θ two-hop graphs, where θ is the number of computing threads.

Since $T_{\{v_i\}}$ can have drastically different sizes with some giant branches becoming stragglers, we adopt the timeout mechanism [36, 41, 46, 60, 61] where a task recursively mining branch T_S can time out after running beyond a time threshold τ_{time} (= 0.1 ms by default), after which any explored node S' during backtracking will become a new task for mining $T_{S'}$. New tasks are directly added back to the task queue to be scheduled for processing by idle threads. Since tasks created from the same top-level branch $T_{\{v_i\}}$ share the same graph g_i, we adopt the approach detailed in Section 6 of [36] to group these tasks into one task group: each initial task with $S = \{v_i\}$ creates a task group that keeps g_i for use by its tasks, while new tasks created due to the decomposition of an existing task in v_i's task group are added to the same task group. A task group keeps track of its active tasks with its own queue, and is released from memory together with g_i when a thread finishes the last task in the group. To keep memory bounded, only a window of θ task groups are processed at any time, and when there are already θ task groups, the next new root-level task $T_{\{v_i\}}$ can only start its evaluation when some existing task group is completed.

Recall that Line 11 of Algorithm 2 removes v_i when $T_{\{v_i\}}$ is processed, and Line 12 subsequently applies CTCP(.) to further shrink the graph G. However, this is not thread-safe in our parallel implementation since it is possible that when a thread is still creating g_i from G that needs a vertex u, another thread may have already finished a different branch $T_{\{v_j\}}$ and deleted u (either because $u = v_j$ or due to CTCP). We, therefore, disable Lines 11–12 but instead prune those vertices v_j with $j < i + 1$ in Lines 7–8 (which is not needed in the serial algorithm due to Line 11) when constructing g_i.

Table 2: Datasets

Dataset	$\|V\|$	$\|E\|$	d_{max}	d_{avg}	$D(G)$	Category
johnson8-4-4	70	1855	53	53	53	Synthetic Graph
keller4	171	9435	124	110	102	Synthetic Graph
socfb-Duke14	9885	506,437	1887	102	85	Facebook Network
ia-wiki-Talk	92,117	360,767	1220	8	58	Interaction Network
soc-buzznet	101,163	2,763,066	64,289	55	153	Social Network
soc-LiveMocha	104,103	2,193,083	2980	42	92	Social Network
soc-gowalla	196,591	950,327	14,730	10	51	Social Network
soc-digg	770,799	5,907,132	17,643	15	236	Social Network
sc-ldoor	909,537	20,770,807	76	46	34	Scientific Computing
soc-youtube-snap	1,134,890	2,987,624	28,754	5	51	Social Network
soc-lastfm	1,191,805	4,519,330	5150	8	70	Social Network
soc-orkut	2,997,166	106,349,209	27,466	71	230	Social Network
wikipedia-link-en	13,593,032	669,183,050	1,052,326	49	1114	Hyperlink Network
dbpedia-link	18,268,991	253,780,418	612,308	14	149	Hyperlink Network
wikipedia-link-en13	25,921,548	1,086,367,222	4,271,341	42	1120	Hyperlink Network
delicious-ui	33,778,221	203,595,714	29,319	6	193	Social Network
soc-sinaweibo	58,655,849	522,642,066	278,489	9	193	Social Network
web-ClueWeb09	147,925,593	893,533,906	308,477	6	192	Web Graph

7 EMPIRICAL STUDIES

This section reports our comprehensive experiments to evaluate the various pruning techniques under the proposed U-MkP framework, as well as its comparison with the state-of-the-art MkP algorithms. Our code is written in C++17 and compiled by g++ version 12.3.0 with optimization flag -O3. All the experiments are conducted on a platform with 24 cores (Intel Xeon Gold 6248R) and 256GB RAM. The unit of time we report is "second" unless stated otherwise.

Datasets and Experimental Settings. We use 26 datasets as described in Appendix N [27]. Due to page limit, we only show the results of 18 graphs as summarized in Table 2, and results on the full datasets can be found in our technical report [27]. In Table 2, d_{max} and d_{avg} indicate the maximum degree and average degree, respectively, and $D(G)$ is the degeneracy. These datasets span a wide range of graph sizes and categories, including 6 with over 10^7 vertices.

We set the time limit as 1800 s (s = seconds). We use × in tables and "OOT" in text descriptions to indicate that execution exceeds this time limit. We tested the datasets for 11 different values of k, i.e., 2, 3, 4, 5, 6, 7, 8, 9, 10, 15 and 20. See Table 10 in our technical report [27] for the MkP sizes of all datasets for all values of k.

Default Configuration of Pruning Techniques: A Summary. Recall all our pruning techniques in Table 1. Interestingly, we find that there exists a configuration of the these techniques (dependent on k only) that leads to the best performance in vast majority of the time: we do not apply upper bounding; for branching, we adopt S-based method when $k \leq 5$ and pivot-based method when $k > 5$; for reduction, we enable Two-Hop, top-level CTCP, RR1–RR3, BR1, BR2, UBR2, but disable recursive CTCP and UBR1. We will report the ablation studies to reach this conclusion next, before we compare this default configuration of U-MkP with other existing algorithms.

Table 3: Comparison of Branching Techniques (Second Best is Underscored)

Dataset	$k = 2$				$k = 3$				$k = 8$				$k = 10$			
	S-Br	R-Br	SR-Br	Pivot-Br	S-Br	R-Br	SR-Br	Pivot-Br	S-Br	R-Br	SR-Br	Pivot-Br	S-Br	R-Br	SR-Br	Pivot-Br
johnson8-4-4	**0.7**	1.1	0.8	1.0	**5.7**	18.9	8.1	10.3	×	×	×	×	×	×	×	**1271.9**
keller4	25.2	49.9	**17.7**	69.8	**28.9**	87.9	31.3	274.0	×	×	×	×	×	×	×	×
socfb-Duke14	**1.2**	1.9	2.1	1.8	**0.8**	1.0	1.7	0.8	656.9	6.2	×	**0.6**	×	52.0	×	**0.6**
ia-wiki-Talk	0.8	1.1	**0.6**	1.3	**0.6**	0.8	0.6	0.8	44.7	2.8	22.8	**0.9**	436.6	4.3	101.0	**1.5**
soc-buzznet	**358.0**	933.6	389.1	1365.3	**189.1**	589.5	770.2	456.0	×	1438.2	×	**75.4**	×	×	×	**150.3**
soc-LiveMocha	3.3	5.6	**2.7**	6.6	**2.2**	3.5	2.6	3.2	492.7	62.2	246.0	**4.1**	×	16.9	388.6	**2.8**
soc-gowalla	0.1	0.1	0.1	**0.1**	0.1	0.1	**0.1**	0.1	**0.1**	0.1	0.1	0.1	0.4	0.5	0.5	**0.1**
soc-digg	**121.9**	226.5	204.9	304.0	**49.4**	162.5	1373.8	80.2	1296.2	×	×	**49.1**	1580.5	×	×	**35.1**
sc-ldoor	14.1	14.7	14.4	**13.6**	24.6	26.4	26.3	**24.2**	**6.9**	7.0	7.2	7.1	9.0	**8.1**	8.8	8.6
soc-youtube-snap	0.4	0.4	0.4	**0.4**	0.5	0.5	**0.5**	0.5	1.4	0.8	1.1	**0.4**	21.5	1.0	9.9	**0.5**
soc-lastfm	3.6	4.6	**2.4**	7.5	**1.9**	2.9	2.1	3.4	3.4	2.0	45.4	**1.1**	54.2	6.8	68.4	**1.6**
soc-orkut	96.1	135.1	179.0	**96.0**	79.1	103.0	511.1	**77.3**	×	×	×	**49.9**	×	×	×	**46.1**
wikipedia-link-en	12.1	12.5	11.8	**11.7**	12.0	**11.9**	14.3	13.4	**67.5**	68.3	69.4	69.0	12.4	12.6	13.5	**12.0**
dbpedia-link	76.0	64.8	**54.8**	233.4	52.2	64.5	**48.2**	404.0	**51.3**	×	53.1	×	**81.1**	×	×	×
wikipedia-link-en13	249.0	260.3	288.5	**240.5**	290.7	277.7	269.8	**247.3**	302.8	304.6	297.7	**284.1**	330.6	283.1	**265.1**	292.3
delicious-ui	62.1	57.8	63.9	**51.9**	99.0	**77.1**	94.1	161.7	1320.7	987.1	**971.4**	×	×	×	×	×
soc-sinaweibo	**141.2**	1036.8	1805.6	306.3	**126.3**	1397.6	×	651.5	×	×	×	×	×	×	×	×
web-ClueWeb09	**43.8**	167.5	155.0	192.0	**70.5**	×	×	×	×	×	×	×	×	×	×	×

Choice of Upper-Bounding (UB) Techniques: Recall that Section 4.1 presented three upper bounding (UB) techniques proposed by the AI community: S-Based, R-Based and SR-Based. We studied the effect of these upper bounding techniques and found that they have a marginal benefit, and in some settings, the cost of upper bound computation itself is quite high and can backfire. For example, the running time on *soc-digg* when $k = 5$ is **102.4** s, 218.5 s, 1266.5 s, and OOT for no UB pruning, S-Based, R-Based, and SR-Based UB pruning, respectively. The full results are shown in Table 11 of our technical report [27], based on which we recommend to apply no bounding technique by default. We do notice an exception that S-Based is quite effective on the two largest graphs, especially *web-ClueWeb09* when $k = 4, 5$. Moreover, S-Based does not bring much slowdown in most cases, so it could be safe alternative for large graphs (e.g., $|V| = 10^7$) and worth trying out if OOT happens.

Choice of Branching Method: Section 4 presented two competitive branching methods: partition-based and pivoting-based, where partition-based methods are based on the above-mentioned 3 UB techniques. Table 3 shows a comparison of these branching methods when $k = 2$, 3, 8 and 10, and the full results are shown in Table 12 of our technical report [27]. The baseline binary branching method (where pivot is selected simply based on the degeneracy order) is not competitive and runs OOT most of the time, so is not included in Table 3 (but is shown in Table 12). Among the 3 partition-based methods, there is no clear winner, but S-based method is the most stable and often performs the best for more time-consuming jobs when k is small, so we adopt S-based branching when $k \leq 5$ by default. On the other hand, the pivoting-based branching method by Maple is a clear winner when k is large, so we adopt pivoting-based branching when $k > 5$ by default. This default scheme of U-MkP generally works very well. For example, as Table 3 shows, when $k = 3$, it took 28.7 s on *keller4* when applying S-based branching but 274.0 s (∼10× slower) when applying pivoting-based branching. On the other hand, when $k = 10$, it did not finish within the time limit on *socfb-Duke14* when applying S-based branching but finished in only 0.6 s (> 3000× faster) when applying pivoting-based branching. Exceptions exist: the advantage of S-based branching goes beyond $k = 5$ on *dbpedia-link* and *delicious-ui*, so can be an alternative to try when OOT happens with pivot-based branching.

Finally, as we shall show in Tables 5–6, kPlexT [35], which uses an improved binary branching method (important for their worst-case time complexity proof), can be a few orders of magnitude slower than our default setting with U-MkP. Since the only difference in pruning techniques between kPlexT and U-MkP is the branching method, it shows that kPlexT's branching method is not competitive in practice where we rarely care about the worst-case performance.

Choice of Reduction Rules: Now, let us consider the 7 reduction rules in Table 1. Section 3 introduced **Two-Hop** and **CTCP** that are applied to prune each top-level subgraph g_i extended from $v_i \in V$. Table 13 in our technical report [27] shows that disabling **Two-Hop** is disastrous and cause most experiments to run OOT, so it should always be enabled. Surprisingly, Table 14 in our technical report [27] shows that disabling **CTCP** at the top level does not cause much slowdown, indicating that its pruning effect is mostly covered also by other techniques. However, since top-level CTCP has a low overhead and is slightly beneficial in most cases, we still enable it by default.

However, we observed that applying CTCP inside the BB(.) procedure (i.e., in reduce_and_prune(.)), as kPlexS [34] does, is expensive. Specifically, Table 15 in our technical report [27] shows that, when enabling CTCP inside BB(.), the execution time was increased by one to two orders of magnitude for many datasets. Thus, U-MkP disables it in BB(.) by default.

Next, we consider the reduction rules presented in Section 5. We tested the efficiency of those rules by disabling each rule at a time and observing the performance difference. Tables 16–18 in our technical report [27] report the effect of **RR1**–**RR3**, and we can see that they can significantly speed up computation, so we enable them by default. Note that conditions of RR1–RR3 are efficient to check.

Reduction rules **BR1** and **BR2** aim to add some vertices directly to S. We report the effect of BR1 and BR2 in Table 19 of our technical report [27], and to our surprise, we can see that they merely make any difference for almost all datasets and values of k, except for *delicious-ui* where a significant speedup is observed. This shows that their conditions seldom hold to allow pruning, but since they are efficient to check, U-MkP enables them by default.

Reduction rule **UBR1** requires access to the number of common neighbors of every pair of vertices in g for condition checking, as is also required by recursive CTCP, so they are usually used together if enabled in BB(.). However, as we have shown in Table 15 in our technical report [27], the overhead of dynamically maintaining these counts is expensive, so U-MkP disables UBR1 by default along with recursive CTCP. On the other hand, we observed that **UBR2** is highly effective on many datasets. Table 4 shows a comparison between our default U-MkP that enables UBR2 and the version that disables it, when $k = 3$, 4, 8 and 10. The full results are shown in Table 20 of our technical report [27]. We can see that enabling UBR2 can

Table 5: Execution Time for Small Values of k (Unit: seconds)

Dataset	$k=2$				$k=3$				$k=4$				$k=5$			
	U-MkP	kPlexT	Maple	DiseMKP	U-MkP	kPlexT	Maple	DiseMKP	U-MkP	kPlexT	Maple	DiseMKP	U-MkP	kPlexT	Maple	DiseMKP
johnson8-4-4	**0.7**	1.7	1.7	1.1	**5.7**	24.0	34.4	6.6	**29.6**	210.9	926.3	239.4	**2.0**	123.0	×	76.1
keller4	25.2	134.6	111.6	**19.7**	**28.9**	2073.1	1251.1	64.0	**1504.8**	×	×	×	**1120.0**	×	×	×
socfb-Duke14	**1.2**	6.4	2.3	129.2	**0.8**	14.4	43.7	×	**1.4**	9.2	30.9	×	**2.6**	21.3	4.8	×
ia-wiki-Talk	0.8	4.1	4.1	**0.7**	**0.6**	2.7	10.1	4.1	**0.4**	0.8	1.3	6.9	**0.7**	0.9	4.1	70.7
soc-buzznet	**358.0**	1577.7	1589.5	×	**189.1**	2744.2	×	×	**153.0**	×	×	×	**1771.0**	×	×	×
soc-LiveMocha	**3.3**	24.6	8.0	8.2	**2.2**	7.8	178.6	19.2	**2.3**	10.5	120.2	120.2	**7.0**	28.1	62.3	×
soc-gowalla	**0.1**	0.1	0.2	1.1	**0.1**	0.1	0.2	0.8	0.1	**0.1**	0.1	2.3	**0.1**	0.3	0.2	8.9
soc-digg	**121.9**	392.2	331.1	×	**49.4**	×	×	×	**33.2**	252.0	×	×	**102.4**	503.7	×	×
sc-ldoor	14.1	**9.9**	11.7	×	24.6	**14.9**	40.8	×	**56.6**	159.9	221.9	×	**71.3**	298.9	572.2	×
soc-youtube-snap	**0.4**	1.3	0.6	1.6	**0.5**	1.4	0.7	2.2	**0.3**	0.4	1.6	2.8	**0.3**	1.0	0.5	10.6
soc-lastfm	**3.6**	8.1	7.9	4.9	**1.9**	6.2	63.9	17.8	**1.5**	6.4	28.8	24.8	**2.2**	7.5	34.1	11.7
soc-orkut	**96.1**	505.2	220.0	×	**79.1**	706.9	603.0	×	**611.5**	×	1047.9	×	×	×	**1473.2**	×
wikipedia-link-en	12.1	**10.7**	65.9	×	12.0	**9.9**	560.5	×	**12.9**	17.6	1485.6	×	**15.3**	44.9	×	1520.1
dbpedia-link	**76.0**	198.4	332.2	×	**52.2**	700.5	×	×	**50.5**	1371.4	×	×	**49.3**	×	×	×
wikipedia-link-en13	**249.0**	321.0	×	×	**290.7**	364.6	×	×	**249.3**	470.5	×	×	**290.9**	552.6	×	×
delicious-ui	**62.1**	109.3	189.6	326.9	**99.0**	441.9	215.1	395.3	**101.2**	1368.2	215.2	394.3	**119.3**	×	×	1452.4
soc-sinaweibo	**141.2**	674.8	512.0	×	**126.3**	×	×	×	**207.3**	×	×	×	**1661.7**	×	×	×
web-ClueWeb09	**43.8**	355.3	224.6	×	**70.5**	×	×	×	**613.2**	×	×	×	×	×	×	×

Table 6: Execution Time for Large Values of k on Real Graphs

Dataset	$k=7$				$k=10$				$k=15$				$k=20$			
	U-MkP	kPlexT	Maple	DiseMKP	U-MkP	kPlexT	Maple	DiseMKP	U-MkP	kPlexT	Maple	DiseMKP	U-MkP	kPlexT	Maple	DiseMKP
johnson8-4-4	×	×	×	×	**1271.9**	×	×	×	**20.4**	36.3	159.5	401.2	0.0	0.0	0.0	**0.0**
keller4	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×
socfb-Duke14	**0.6**	307.8	6.1	×	**0.6**	×	9.0	×	**2.7**	×	18.3	×	**1.7**	×	25.2	×
ia-wiki-Talk	**0.5**	4.7	0.6	×	**1.5**	×	2.5	×	**3.6**	×	25.5	×	**23.7**	×	×	×
soc-buzznet	**76.1**	×	×	×	**150.3**	×	×	×	**520.7**	×	×	×	**701.7**	×	×	×
soc-LiveMocha	**4.1**	495.9	365.5	×	**2.8**	×	24.9	×	**0.9**	×	1.9	×	**3.2**	×	335.7	×
soc-gowalla	**0.1**	0.4	0.1	15.1	**0.1**	66.5	0.1	×	**0.2**	×	0.6	×	**0.4**	×	4.5	125.0
soc-digg	**54.5**	1529.0	1683.5	×	**35.1**	×	608.2	×	**7.9**	×	28.1	×	**7.4**	×	26.5	×
sc-ldoor	**11.3**	29.8	16.3	×	8.6	**5.7**	5.7	×	7.3	**3.0**	4.3	×	10.1	**5.2**	6.5	×
soc-youtube-snap	**0.3**	1.5	0.5	186.4	**0.5**	143.8	0.5	×	**1.2**	×	3.6	×	**4.6**	×	1347.6	×
soc-lastfm	**3.5**	168.8	7.0	534.1	1.6	2014.8	**1.3**	2958.2	**5.6**	×	6.7	×	**21.3**	×	824.1	×
soc-orkut	**68.8**	×	1200.9	×	**46.1**	659.4	195.5	×	**33.6**	519.9	142.5	×	**37.2**	×	98.3	×
wikipedia-link-en	**247.1**	580.8	×	×	**12.0**	179.5	1639.1	39.1	9.9	**8.4**	18.7	101.7	**7.2**	8.9	15.3	14.4
dbpedia-link	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×
wikipedia-link-en13	**304.4**	652.2	×	×	**292.3**	×	×	×	**297.2**	×	×	×	**277.6**	×	×	×
delicious-ui	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×
soc-sinaweibo	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×
web-ClueWeb09	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×

Table 4: Effect of Reduction Rule UBR2

Dataset	$k=3$		$k=4$		$k=8$		$k=10$	
	w/ UBR2	w/o UBR2	w/ UBR2	w/o UBR2	w/ UBR2	w/o UBR2	w/ UBR2	w/o UBR2
johnson8-4-4	**5.7**	7.1	**29.6**	74.4	×	×	**1271.9**	×
keller4	**28.9**	43.5	**1504.8**	×	×	×	×	×
socfb-Duke14	**0.8**	3.7	**1.4**	14.1	**0.6**	×	**0.6**	×
ia-wiki-Talk	**0.6**	1.4	**0.4**	1.4	**0.9**	×	**1.5**	×
soc-buzznet	**189.1**	×	**153.0**	1701.1	**75.4**	×	**150.3**	×
soc-LiveMocha	**2.2**	6.7	**2.3**	7.5	**4.1**	×	**2.8**	×
soc-gowalla	0.1	**0.1**	0.1	**0.1**	0.1	**0.1**	**0.1**	908.4
soc-digg	**49.4**	558.3	**33.2**	918.2	**49.1**	×	**35.1**	×
sc-ldoor	**24.6**	25.2	**56.6**	64.6	**7.1**	7.3	8.6	**8.4**
soc-youtube-snap	**0.5**	0.6	**0.3**	0.4	**0.4**	×	**0.5**	×
soc-lastfm	**1.9**	7.7	**1.5**	5.3	**1.1**	×	**1.6**	×
soc-orkut	**79.1**	809.8	**611.5**	×	**49.9**	×	**46.1**	×
wikipedia-link-en	**12.0**	16.1	**12.9**	468.6	**69.0**	×	**12.0**	×
dbpedia-link	**52.2**	68.6	**50.5**	50.8	×	×	×	×
wikipedia-link-en13	290.7	**229.0**	249.3	**217.6**	284.1	**247.2**	292.3	**284.5**
delicious-ui	**99.0**	168.2	**101.2**	171.8	×	×	×	×
soc-sinaweibo	**126.3**	×	**207.3**	×	×	×	×	×
web-ClueWeb09	**70.5**	×	**613.2**	×	×	×	×	×

significantly speed up computation. For example, on *socfb-Duke14*, U-MkP w/o UBR2 cannot finish within 1800 s when $k = 8$, but with UBR2, U-MkP finishes in 0.6 s, so the speedup is 3000×.

Comparison of MkP Algorithms. Recall from the algorithm dominance graph shown in Figure 4 that among existing MkP algorithms, only kPlexT [35], Maple [53] and DiseMKP [44] are competitive baselines (Seesaw has no code released). We, therefore, select these three baselines to compare with our U-MkP framework with the previously mentioned default configuration of pruning rules. Table 5 shows the running time of the compared algorithms when $k = 2$, 3, 4 and 5 (so U-MkP adopts S-based branching), where we can see that U-MkP performs the best in vast majority of the datasets and often beats the second best by many times to over an order of magnitude for time-consuming jobs. Even when U-MkP is not the best, it is very close to the best (except for the only case of soc-orkut when $k = 5$ where Maple wins but is not too far from OOT). For example, U-MkP is 38.5× faster than DiseMKP (the second best) on *johnson8-4-4* when $k = 5$, and only 1.28× slower than the winner algorithm for *keller4* when $k = 2$. Among the other algorithms, there is no clear second best, but kPlexT seems to win on more datasets. However, due to its ineffective binary branching method, kPlexT is still much slower than U-MkP. Maple is slow in most time, showing that its pivot-based branching method is not effective for small k values. Finally, although DiseMKP uses S-based branching, it does not utilize effective reduction rules like UBR2, so is not competitive.

Table 6 shows the running time of the compared algorithms when $k = 7$, 10, 15 and 20 (so U-MkP adopts pivot-based branching), where we can see that U-MkP still performs the best and often beats the second best by a few orders of magnitude. Maple is the only other algorithm that can properly handle large k values, thanks to its pivot-based branching. However, it can still be a few orders of magnitude slower than U-MkP since it does not utilize effective reduction rules like UBR2. Among the other two algorithms, DiseMKP simply cannot handle large values of k and runs OOT on most datasets; while even though kPlexT is the latest algorithm that proposed UBR2, it generally cannot handle the cases when $k = 15, 20$ due to its suboptimal approach for binary branching.

While Tables 5 and 6 cover only a subset of k on 18 datasets, Table 21 of our technical report [27] shows results on all 26 datasets for all values of k, where we can see that U-MkP is a clear winner.

Performance of MkP Variants. Section 6 presented a two-phase approach to compute all MkPs as well as the densest MkP. Table 7 reports the results of running this variant for $k = 5$ on 18 graphs (full results on all our 26 graphs and all k values are shown in Table 22

Table 8: Running Time of Parallel U-MkP on Representative Datasets with Varying Number of Threads

Dataset	$k=2$						$k=3$						$k=4$						$k=5$					
	1	2	4	8	16	32	1	2	4	8	16	32	1	2	4	8	16	32	1	2	4	8	16	32
keller4	27.6	13.9	6.7	3.4	1.7	0.9	247.0	120.3	60.7	30.7	16.0	8.0	1480.2	1028.2	728.3	423.0	182.1	60.1	1120.2	620.4	287.5	120.3	85.3	43.2
soc-buzznet	301.6	150.5	74.9	36.7	19.1	9.4	145.7	73.3	36.5	18.0	9.2	4.7	143.4	71.1	35.0	17.9	9.0	4.8	1720.7	824.2	421.2	231.4	134.2	61.2
soc-LiveMocha	2.4	1.2	0.6	0.3	0.2	0.1	1.5	0.7	0.4	0.2	0.1	0.1	3.7	1.7	0.8	0.4	0.2	0.2	18.8	10.1	5.5	3.0	1.5	1.5
soc-digg	91.6	45.9	22.2	11.5	5.7	2.9	29.5	15.2	7.6	3.7	1.9	1.0	42.5	16.0	6.6	3.0	1.3	0.9	100.2	54.3	27.3	14.9	8.2	4.3
sc-ldoor	5.9	3.0	1.6	0.8	0.5	0.7	24.2	12.3	5.8	3.7	2.8	3.2	48.2	24.3	15.2	8.9	4.3	3.2	72.8	38.2	20.9	10.3	6.7	3.2
delicious-ui	18.8	9.9	5.3	2.7	1.4	0.6	95.7	48.4	25.4	13.3	6.8	2.8	345.2	178.7	94.8	49.5	24.9	10.2	1282.2	834.2	612.1	406.9	205.2	88.4
soc-sinaweibo	32.0	18.5	10.3	5.3	3.6	2.6	30.2	18.1	10.1	5.2	3.7	2.6	60.1	34.1	19.0	8.7	6.2	5.1	1812.2	918.3	345.2	121.8	74.3	45.7
web-ClueWeb09	9.2	5.9	3.2	1.7	1.2	1.0	16.2	8.8	4.8	2.4	1.9	1.6	57.2	29.4	15.8	7.8	5.7	4.8	52.2	28.9	17.4	8.4	4.9	3.7

Table 7: Finding All MkPs and Densest MkP

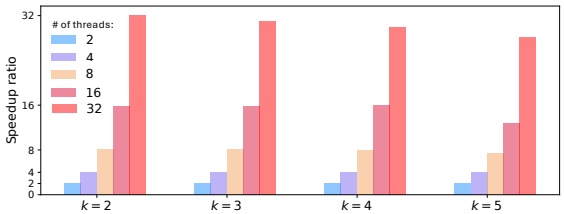
Dataset	$k=5$			
	Time	#MkPs	$\lvert E_{\text{Phase 1}}\rvert$	$\lvert E_{\text{Densest}}\rvert$
johnson8-4-4	9	226	644	644
keller4	×	×	×	×
socfb-Duke14	12	75	2166	2172
ia-wiki-Talk	3	95	530	534
soc-buzznet	×	×	×	×
soc-LiveMocha	15	24	664	676
soc-gowalla	0	40	972	980
soc-digg	90	15	4976	4986
sc-ldoor	×	×	×	×
soc-youtube-snap	1	8	584	590
soc-lastfm	3	2	622	622
soc-orkut	×	×	×	×
wikipedia-link-en	31	24,837	1,219,880	1,219,880
dbpedia-link	91	2	2776	2776
wikipedia-link-en13	67	1	186170	186,170
delicious-ui	882	23	110	132
soc-sinaweibo	×	×	×	×
web-ClueWeb09	×	×	×	×

in our technical report [27]), including (1) the running time, (2) the number of MkPs found, (3) the number of vertices and the number of edges in the MkP found by Phase 1, and (4) the number of edges of the densest MkP found in Phase 2. We can see that some graphs have many MkPs, so finding one of them is not sufficient to catch all dense communities. While *wikipedia-link-en* has 24,837 MkPs so are not selective, many graphs have several to tens of MkPs which are reasonably selective and interesting for users to examine all these structures. We can also see that MkPs found in Phase 1 are not the densest on many graphs. For example, on *soc-LiveMocha*, Phase 1 finds an MkP with 664 edges but the densest has 676 edges.

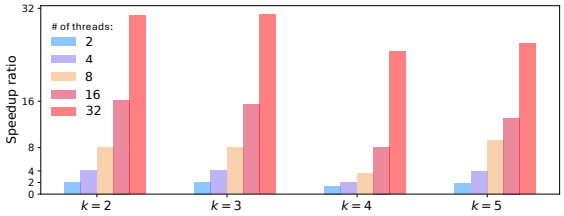
Performance of U-MkP Parallelization. We implemented the parallel version of U-MkP based on the description in Section 6. Table 8 reports the running time on 8 datasets when using 1, 2, 4, 8, 16 and 32 threads, respectively, where we use the default timeout threshold $\tau_{time} = 0.1$ ms for load balancing (which consistently works well). We chose representative datasets where the job time in serial execution is at least 10 seconds when $k = 5$, so that it is worth for parallelization. As Table 8 shows, our parallel algorithm is efficient and scales up well with the number of CPU cores on all the datasets. We also show the speedup ratio on 2 representative datasets in Figure 10. We can see that our parallelization can achieve a speedup ratio of up to 28.1× with 32 threads. The complete results for all datasets can be found in Table 23 of our technical report [27].

8 RELATED WORK

This paper has surveyed the algorithms for **maximum** k-plex finding as summarized in Table 1. There are also works for finding **maximal** k-plexes (i.e., those without a supergraph that is also a k-plex) [31, 36, 38, 39, 54, 64], often with a size threshold q to find only those with at least q vertices. Among them, ListPlex [54] proposes to create initial tasks each of which consists of a top-level vertex v_i and a subset of its two-hop neighbors, and extend these vertices



(a) *soc-buzznet*



(b) *keller4*

Figure 10: Speedup Ratio of Parallel U-MkP

with candidates from v_i's one-hop neighbors. This approach not only reduces the worst-case time complexity [54], but is also efficient and hence adopted by a later work [36]. The authors of ListPlex aim to apply this idea to the MkP problem by proposing Maple [53], but we found that their implementation does not follow their paper description. In general, finding maximal k-plexes is more expensive than MkP since the size lower bound remains at q rather than $|P| + 1$, so many pruning techniques are not as effective; finding maximal k-plexes also needs to avoid emitting non-maximal results with the help of an exclusion set following the Bron-Kerbosch algorithm [32].

9 CONCLUSION

We proposed U-MkP, a framework for finding a maximum k-plex that can be adapted to find all maximum k-plexes or the one with the most edges. Our framework can integrate the various pruning techniques from nine recent algorithms including kPlexT, Maple, Seesaw, DiseMKP, kPlexS, KpLeX, Maplex, BnB and BS, which were summarized into three categories: those for (1) branching, (2) upper bounding, and (3) reduction during subgraph exploration. We found that different pruning techniques can have drastically different performance impacts, and obtained interesting new insights about these techniques not studied by prior works. Moreover, we found that there exists a configuration of the techniques dependent on k that leads to the best performance in vast majority of the time, which we recommended as the default configuration of U-MkP.

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[7] keller4. https://networkrepository.com/keller4.php.
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