LINE GRAPH OF GAMMA-ACYCLIC DATABASE SCHEMES AND ITS RECOGNITION ALGORITHM

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Abstract

In this paper we describe the properties of the line graph of γ-acyclic hypergraphs. Based on the properties, an efficient algorithm is given for determining whether a hypergraph is γ-acyclic. The algorithm runs in $O(n(n+e))$ time for a hypergraph with its line graph having n vertices and e edges.

1. Introduction

There is a natural correspondence between database schemes and hypergraphs. A number of basic desirable properties of database schemes have been shown to be equivalent to acyclicity (2) . Further R. Fagin has recently defined two types acyclicity for hypergraphs which he calls β-acyclicity and γ-acyclicity (3) (where the early type of acyclicity he calls α-acyclicity). He proves that γ-acyclicity is equivalent to some more desirable conditions involving monotone-increasing joins and unique relationship among attrbutes, which are not equivalent to other acyclicities.

There have been polynomial-time algorithms for determining α-acyclicity (4,2), β-acyclicity (3) and γ-acyclicity (5,3), respectively. A linear-time algorithm for α-acyclicity has recently been given by Tarjan and Yannakakis (6). We also note that Karen Chase (7) offers two methods to make a α-cyclic hypergraph to be α-acyclic. The purpose of this paper is to discuss the properties of the line graph corresponding to γ-acyclic hypergraph. Based on the properties an efficient algorithm is given for determining whether a hypergraph is γ-acyclic by means of the line graph of the hypergraph.

2. γ-acyclicity

A hypergraph (1) is a pair (N, E), where N is a finite set of nodes and E is a set of edges which are arbitrary nonempty subset of N.

A γ-cycle in a hypergraph H is a sequence

$(S_1, x_1, S_2, x_2, \ldots, S_m, x_m, S_{m+1})$

such that

(a) $x_1, \ldots, x_m$ are distinct nodes of H:
(b) $S_1, \ldots, S_m$ are distinct edges of H and $S_{m+1} = S_1$;
(c) $m \geq 3$, that is, there are at least 3 edges involved;
(d) x_i is in S_i and S_{i+1} ($1 \leq i \leq m$); and
(e) if $1 \leq i < m$, then x_i is no S_j except S_i and S_{i+1} .

where m is the size of γ-cycle.

Definition 1 A hypergraph is γ-cyclic if it has a γ-cycle.

Let $(S_1, \ldots, S_m, S_{m+1})$ be a sequence of sets, where $S_1, \ldots, S_m$ are distinct and $S_{m+1} = S_1$. Let us call S_i and S_{i+1} neighbors ($1 \leq i \leq m$); note, in particular, that S_m and S_1 are neighbors. Let us call $(S_1, \ldots, S_m, S_{m+1})$ a pure cycle if $m \geq 3$ and if whenever $i \neq j$, then $S_i \cap S_j$ is nonempty if and only if S_i and S_j are neighbors.

Definition 2 A hypergraph is γ-cyclic if it has either a γ-cycle of size 3 or a pure cycle.

Lemma 1 Definition 1-2 of γ-cyclicity are equivalent (3).

A hypergraph is γ-acyclic if it is not γ-cyclic. It is easy to see that a hypergraph is γ-cyclic according to Definition 2 if it contains at least one of two kinds of "forbidden configurations" as shown in Figure 1.

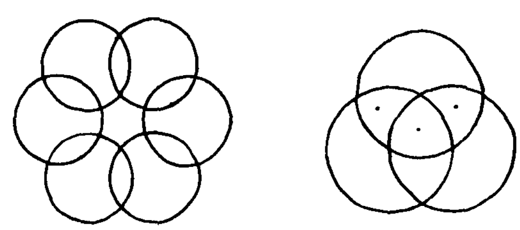

Figure 1

3. A Characterization of Line Graph of a Hypergraph

Let $H=(N;X_1,X_2,\ldots,X_n)$ be a hypergraph with n edges. The line graph of H is defined to be the simple graph L(H) of order n whose vertices $x_1,x_2,\ldots,x_n$ respectively represent the edges $X_1,X_2,\ldots,X_n$ of H and with vertices x_i and x_j joined by an edge if and only if $X_i \cap X_j = \phi$. We say that W_{ij} is the weight of the edge between x_i and x_j in L(H) if $W_{ij}=X_i \cap X_j$ and $W_{ij} \neq \phi$. Figure 2 shows a hypergraph, its corresponding line graph and weights.

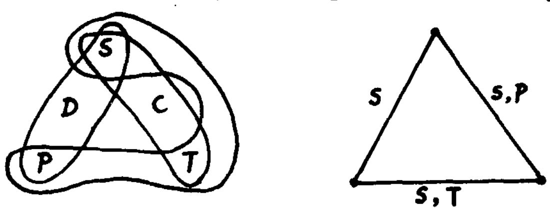

Figure 2

A graph G is a chordal graph if every cycle in G with at least four distinct vertices has a chord. There are many important results about chordal graph (8).

Lemma 2 Line graph L(H) of a γ-acyclic hypergraph H is chordal.

Proof. Let H be a γ-acyclic hypergraph, and L(H) be its line graph. Assume that L(H) is not chordal. We shall show that H is γ-cyclic.

Since L(H) is not chordal, there is a cycle $(x_1,e_1,\ldots,x_k,e_k,x_{k+1})$ in L(H) (if no cycle, then L(H) must be chordal), such that

(a) $x_1,x_2,\ldots,x_k$ are distinct vertices and $x_{k+1}=x_1$;
(b) $e_1,e_2,\ldots,e_k$ are distinct edges;
(c) $k \geq 4$; and
(d) there are no edges in L(H) connecting two vertices of the cycle, except $e_1,e_2,\ldots,e_k$.

It follows immediately that there is a cycle $(S_1,S_2,\ldots,S_k,S_{k+1})$ in H corresponding to $x_1,x_2,\ldots,x_k,x_{k+1}$ (note where $S_{k+1}=S_1$). It is easy to see that $S_i \cap S_j = \phi$ if and only if S_i and S_j are neighbors. Together with $k \geq 4$, we claim the cycle in H is a pure cycle. From Definition 2, H is γ-cyclic. This is a contradiction. Thus Lemma 2 is proved.

Now, let us see the case of the cycle with size 3 in a hypergraph. Let us say that a hypergraph H is pairwise nondisjoint if every pair of edges in H is nondisjoint. Let us call a complete graph K_3 isosceles triangle if there are at least two edges with precisely the same weight in K_3.

Lemma 3 If a pairwise nondisjoint hypergraph H with 3 edges is γ-acyclic, then its L(H) is an isosceles triangle.

Proof. Let H be a pairwise nondisjoint hypergraph with 3 edges, then it is obvious that L(H) is a triangle.

Assume L(H) is not isosceles. Let A, B and C be vertices of L(H), W_1,W_2 and W_3 be weights of the edges in L(H), respectively, as shown in Figure 3.

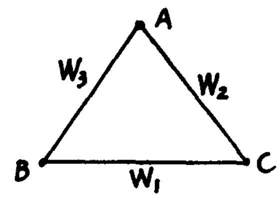

Figure 3

By assumption, we know that $W_1 \neq W_2$, $W_2 \neq W_3$ and $W_1 \neq W_3$. Now we show that none of the followings is true.

$$W_1=W_2 \cup W_3 \quad (3.1)$$
$$W_2=W_1 \cup W_3 \quad (3.2)$$
$$W_3=W_1 \cup W_2 \quad (3.3)$$

Because of symmetry, we only need to deny (3.1). Since by (3.1), $W_2 \subseteq W_1$ and $W_3 \subseteq W_1$, then for each $a \in W_2$, it follows $a \in W_1$, or $a \in B$. On other hand, $a \in W_2$, that is, $a \in A$. Therefore $a \in A \cap B$, or $a \in W_3$. So $W_2 \subseteq W_3$. Similarly, for each $b \in W_3$, we obtain that $b \in A$, $b \in W_1$ and $b \in C$. From $b \in A \cap C$ or $b \in W_2$, we claim $W_3 \subseteq W_2$, thus $W_2=W_3$. The contradiction shows that none of those equations holds.

Now we shall prove that at least two following equations are true.

$$W_1 - W_2 \cup W_3 \neq \phi \quad (3.4)$$
$$W_2 - W_1 \cup W_3 \neq \phi \quad (3.5)$$
$$W_3 - W_1 \cup W_2 \neq \phi \quad (3.6)$$

We can assume without loss of generality that neither (3.4) nor (3.5) is true, that is, $W_1 - W_2 \cup W_3 = \phi$ and $W_2 - W_1 \cup W_3 = \phi$. Based on (3.1) and (3.2), we obtain

$$W_1 \subset W_2 \cup W_3 \quad (3.7)$$
$$W_2 \subset W_1 \cup W_3 \quad (3.8)$$

Let $F=W_1 \cap W_2$, $F_1=W_1-F$ and $F_2=W_2-F$ (note, there is at least one of F_1 and F_2 being nonempty, otherwise $W_1=W_2=F$). There are three cases as follows:

Case 1. F_1 is nonempty. By (3.7), $F_1 \cup F \subset F_2 \cup F \cup W_3$, i.e., $F_1 \subset F_2 \cup W_3$, so $F_1 \subset W_3$ (because $F_1 \cap F_2 = \phi$). It follows $F_1 \subset A$. On the other hand, $F_1 \subset W_1$, i.e., $F_1 \subset C$. Hence, $F_1 \subset A \cap C$, i.e., $F_1 \subset W_2$, it is impossible;

Case 2. F_2 is nonempty. Similar to Case 1, by (3.8), we can obtain $F_2 \subset W_1$, it is also impossible;

Case 3. Both F_1 and F_2 are nonempty.
Similar to Case 1 and Case 2.

Thus there exist distinct nodes a and b, such that

$$a \in W_1 - W_2 \cup W_3$$
$$b \in W_2 - W_1 \cup W_3$$

We can also select a node $c \in W_3$ (of course, $a \neq c$, $b \neq c$). It is clear that (A,a, B,b,C,c,A) is a γ-cycle. This is the desired contradiction which proves Lemma 3.

In fact, the converse to Lemma 3 is also true. Let us see the following theorem by which an efficient algorithm is given later.

Theorem 1 A hypergraph H is γ-acyclic if and only if its L(H) is chordal and every triangle in L(H) is isosceles.

Proof. ($\Rightarrow$): Let H be a γ-acyclic hypergraph, and L(H) be the line graph of H. By Lemma 2, L(H) is chordal. For every three pairwise nondisjoint edges in H, since they are not γ-cycle, their corresponding triangle in L(H) is isosceles.

($\Leftarrow$): Let L(H) corresponding to H be a chordal graph with every triangle being isosceles. We must show that H is γ-acyclic. Since L(H) is chordal, then H has no pure cycle with size more than 3.

Consider every three pairwise nondisjoint edges (S_i, S_j, S_k) in H which corresponding to every isosceles triangle in L(H), respectively. Assume without loss of generality $S_i \cap S_j = S_i \cap S_k$, then no matter how we select $x \in S_i \cap S_k$, there exist $x \in S_i \cap S_j$, i.e., $x \in S_j$. Similary, for each $y \in S_i \cap S_j$, we obtain $y \in S_i \cap S_k$, i.e., $y \in S_k$.

Therefore there is no γ-cycle of size 3 in H. Of course, there is no pure cycle of size 3 in H. Thus H is γ-acyclic.

It is known that for some graph G, there is no graph whose line graph is G. A ordinary undirected graph (without self-loops) is, of course, a hypergraph whose each edges has only two or one node For example, there is no graph whose line graph is the graph shown in Figure 4.

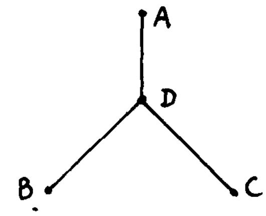

Figure 4

Here, we only consider the graphs which are line graphs of given hypergraphs.

4. An Efficient Algorithm for Testing γ-acyclicity

Based on the properties of line graph of γ-acyclic hypergraph, we apply the following operations sequentially to hypergraph H.

(a) Transform H into its L(H);
(b) Check whether L(H) is chordal, if it is not, then H is γ-cyclic;
(c) Check whether every triangle in L(H) is isosceles, if one of triangle is not isosceles, then H is γ-cyclic; otherwise H is γ-acyclic.

The algorithm is shown in Figure 5.

```
Algorithm 1
BOOLEAN PROCEDRUE HYPERGRAPH(H);
  BEGIN   L←Φ;
          FOR each pair hyperedges Ei, Ej DO
            IF Ei∧Ej≠Φ THEN BEGIN Wij← Ei ∧ Ej ;
                                  add Wij to graph L as an edge;
                                  COMMENT Ep and Wpq are bit string
                                    representing the nodes of H;
                            END;
          IF L is not a chordal graph THEN RETURN FALSE;
          FOR each edge eij of L DO
            FOR each vertex xk of L DO
              IF (xi,xj,xk) is in triangle Tijk THEN
                IF Tijk is not isosceles THEN RETURN FALSE;
          RETURN TRUE;
  END.
```

Figure 5

The following Lemma is important for our algorithm.

Lemma 4 A chordal graph recognition may be performed in O(n+e) time, where n is the number of vertices and e is the number of edges in the graph (9,10).

Theorem 2 Algorithm 1 is correct and runs in O(n(n+e)) time, for L(H) having n vertices and e edges.

Proof. By Theorem 1, Algorithm 1 is correct.

Let us analyse the complexity of the algorithm. First we consider the time spent in forming L(H). The maximal number of the edges in L(H) is $O(n^2)$. The operation for tesing chordal graph should make our algorithm run in O(n+e) time according to Lemma 4. Finally the time spent in checking every isosceles triangle is $O(e\cdot n)$. Thus, the bound is

$$O(n^2+n+e+e\cdot n)=O(n(n+e))$$

The proof is complete.

However, the details of the algorithm may be improved, such that the algorithm would be more efficient. One thing should be emphasized here, the input of the algorithm is a matrix of a hypergraph, which may be called representative matrix.

5. Conclusion

We have discussed the properties of line graphs corresponding to hypergraphs. Those properties are also interesting graph-theoretic facts.

However, how to transform the cyclic database scheme into the γ-acyclic is still the problem to be solved. Another question to be settled is whether there exist a linear-time algorithm for testing γ-acyclicity.

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References

(1) Berge, C. "Graphs and hypergraphs", English Trans. By Edward Minieka, North Holland, 1973.

(2) Beeri, C., Fagin, R., Maier, D. and Yannakakis, M. "On the desirability of database schemes", J. ACM 30, 3 (July 1983).

(3) Fagin, R. "Degrees of acyclicity for hypergraphs and relational database schemes", J. ACM 30, 3 (July 1983).

(4) Graham, M. H. "On the universal relation", Tech. Rep., Univ. of Toronto, Ont., Can., Sept. 1979.

(5) D'Atri, A., et al. "Acyclic hypergraphs: Their recognition and top-down versus bottom-up generation", Tech. Rep. R. 29, Consiglio Nazionale Delle Richerche, Istituto di Analisi dei Sistemi ed Informatica, 1982.

(6) Tarjan, R. E. et al. "Simple linear-time algorithm to test chordality of graphs, test acyclicity of hypergraphs, and selectively reduce acyclic hypergraphs", Tech. Rep. Bell Labs. Murray Hill, N. J., May. 1982.

(7) Karen Chase "Join graph and acyclic database schemes", Proceedings of the Seventh International Conference on VLDB, 1981.

(8) Gavril, F. "Algorithms for minimum coloring, maximum clique, minimum covering by cliques and maximum independent set of a chordal graph", SIAM J. Comput. 1 (1972).

(9) Lueker, G. S. "Efficient algorithm for chordal graphs and interval graphs", Ph. D. Diss., Prog. Appl. Math. and Dept. EE, Princeton U. 1975.

(10) Rose, D. J., Tarjan, R. E., Lueker, G. S. "Algorithmic aspects of vertex elimination on graphs", SIAM J. Comput. 5, 2 (1976).